## THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

At the meeting of the AMERICAN MATHEMATICAL SOCIETY held on May 29, 1897, the By-Laws of the Society were amended to provide that regular meetings should thereafter be held on the last Saturdays of October, February and April, unless otherwise ordered by the Council. The Annual Meeting, held in the last week of December, is unaffected by this action. The reduction of the number of New York meetings to four annually, instead of eight as heretofore, was made with a view to securing greater prominence and interest for each meeting, and to affording the members of the Society a better opportunity for scientific and social intercourse. For this purpose, and in compensation for the reduction in the number of meetings, it was provided that each meeting should hereafter extend through two sessions, to be held in the morning and afternoon. The wisdom of the new departure was abundantly demonstrated at the first meeting held under it on Saturday, October 30, 1897, in New York City.

Forty-one persons were in attendance, of whom thirty-seven were members of the Society. Nine papers were presented at the two sessions. Both numbers greatly surpass those of any previous regular meeting, and the result is especially gratifying in view of the brief interval since the highly successful Summer Meeting of the Society at Toronto.

In the absence of President Newcomb, the Vice-President, Professor R. S. Woodward, occupied the chair. The register of members in attendance was as follows: Professor Cleveland Abbe, Professor E. W. Brown, Dr. J. B. Chittenden, Professor F. N. Cole, Professor E. S. Crawley, Dr. J. W. Davis, Professor T. W. Edmondson, Professor A. M. Ely, Professor T. S. Fiske, Mr. P. R. Heyl, Dr. G. W. Hill, Dr. J. E. Hill, Professor H. Jacoby, Mr. S. A. Joffe, Mr. C. J. Keyser, Professor P. Ladue, Professor G. Legras, Dr. G. H. Ling, Dr. E.McClintock, Mr. J. Maclay, Professor H. P. Manning, Professor M. Merriman, Professor G. D. Olds, Mr. J. C. Pfister, Professor M. I. Pupin, Professor J. P. Pierpont, Professor J. K. Rees, Mr. C. H. Rockwell, Professor C. A. Scott, Professor H. Taber, Professor H. D. Thompson, Professor C. L. Thornburg, Professor J. M. Van Vleck, Professor E. B. Van Vleck, Professor J. B. Webb, Mr. G. L. Wiley, Professor R. S. Woodward.

The Council announced the election of the following persons to membership in the Society: Mr. Paul Capron, Dummer Academy, South Byfield, Mass.; Mr. John Kinsey Gore, Prudential Insurance Company, Newark, N. J.; Professor Alfred George Greenhill, Artillery College, Woolwich, England. Ten applications for membership were reported.

In the interval between the sessions a large number of the members took advantage of the opportunity to lunch together. The same convenient arrangement will be afforded at all future meetings in New York.

The following papers were presented:—

(1) Dr. G. W. Hill: "Intermediary orbits in the lunar theory."

(2) Mr. P. R. Heyl: "Notes on the theory of light on the hypothesis of a fourth dimension."

(3) Professor C. A. Scott: "Note on linear systems of curves."

(4) Professor R. S. Woodward: "On the cubic equation defining the Laplacian envelope of the earth's atmosphere."

(5) Professor R. S. Woodward: "On the integration of a system of simultaneous linear differential equations."

(6) Professor Mansfield Merriman: "The probability of hit on a target when the probable error in aim is known; with a comparison of the probabilities of hit by the methods of independent and parallel fire from mortar batteries."

(7) Professor E. W. Brown: "Note on the steering of an eight-oared boat."

(8) Professor A. S. Chessin: "Note on hyperelliptic integrals."

(9) Dr. E. O. Lovett: "Note on the fundamental theorems of Lie's transformation groups."

Dr. Hill's paper, which will be published in the *Astronomical Journal*, discusses certain modern refinements in the planetary theory. The difficulty astronomers have met in endeavoring to push the approximation to the coördinates of a planet beyond the simple Keplerian theory, in that the time appeared as a multiplier outside of the trigonometrical functions, is at once removed if we divide the potential function otherwise than is commonly done. This modification has given rise to what are known as *intermediary orbits*. But the mode in which Gyldén and others have introduced the latter seems unnecessarily obscure, as they are often discussed without the slightest reference to the law of gravitation. In the lunar theory much will be gained if we can make the perigee and node have a uniform motion

already in the first approximation. Now the disturbing function of the solar action may be considered as involving a term rigorously proportional to the square of the radius and another proportional to the square of the moon's perpendicular distance from the plane of the ecliptic. The first will produce a motion of the perigee while the second does the same for the node.

The integration of the equations of motion naturally brings in the $\theta$-function of Jacobi. But for this it is better to substitute its periodic development, as tables of elliptic functions are necessarily to double entry and are consequently difficult in interpolation; moreover, they do not readily adapt themselves to processes of combination as the circular functions do. In all special cases, where the values of the arbitrary constants are involved, it is always much shorter and easier to follow methods numerical in character.

The essential features of Mr. Heyl's paper may be summarized as follows: Statement of the 4-space theory of light; 3-space analogy—dust on a drumhead. Differential equation for a solid vibrating in a 4th dimension. Identity of this equation with the equation for sound; the difference one of interpretation of the dependent variable. Mutually supplementary character of the two interpretations illustrated by a consideration of boundary conditions. Problems in rectangular coördinates; problems in hypercylindrical coördinates. The functions $\frac{\sin mr}{r}$ and $\frac{\cos mr}{r}$ and the development of functions in terms of these as modified Fourier's series and Fourier's integrals. Slight departure from the law of inverse squares. Curious apparent retroactive effect in the æther.

Professor Scott's paper deals with two questions that present themselves when a twofold linear system of curves, a *net*, is used for the rational transformation of the plane. Divergence from the usual relation of points and curves is exhibited when the net $(\varphi_1\varphi_2\varphi_3)$ has a fundamental curve (*i. e.*, a curve met by the general curve of the net only at the fixed points) represented by a common factor $B$ in two distinct $\varphi$'s, and the modified form of the usual relation depends upon the power to which this factor occurs. The determination of the highest power of $B$ that can occur without any specialization in the position of the fundamental points is the principal object of the paper; the theorem is proved that the $k+1$ determining curves of the $k$-fold system

from which the net is derived by imposing extrinsic conditions can be written in a certain order, viz.: *one* not containing $B$, *two* containing $B^1$, *three* containing $B^2$, etc.; and that for the determination of the net the first of these, any other one, and any other before a certain one, may be chosen without affecting the fundamental points. Divergence from the usual relation occurs also when one or more of the $\varphi$'s have at one or more of the fundamental points a multiplicity greater than that assigned; and the determination of the limits within which this irregularity can occur without specialization in the fundamental points is the second matter considered.

In Chapter VII. of vol. 2 of the Mécanique Céleste Laplace determines from mechanical principles a limiting envelope to the atmosphere of a rotating planet. This envelope is symmetrical with respect to the axis of rotation of the planet, and the curve of intersection of a meridian plane with the envelope is defined by the following cubic equation:

$$x^3 - \frac{2}{a x_0 \sin^2\theta} x + \frac{2}{a \sin^2\theta} = 0,$$

where $x$ is the ratio of the radius vector, measured from the planet's center, to the radius of the planet; $\theta$ is the polar distance of the radius vector; $a$ is the ratio of centrifugal acceleration to the acceleration of gravity along the equator of the planet; and $x_0$ is the value of $x$ for $\theta = 0$.

Professor Woodward's first paper alludes first to the interesting way in which Laplace discussed this equation without the aid of modern methods; secondly, to the ease with which the least and greatest roots of the equation are discovered from mechanical considerations; thirdly, to the elegant solution of the equation afforded by the goniometric method, which indicates at once that one of the three real roots applicable to the envelope; and fourthly, to the special fitness of the latter method in supplying for the root used an expression which it is practicable to apply in computing the volume, mass, and mass distribution of the terrestrial atmosphere.

Professor Woodward's second paper will be published in the *Astronomical Journal*. The differential equations considered in this paper are the special forms of the equations of rotation of a non-rigid mass subject to no external forces and to the condition that its principal moments of inertia are equal. The equations are

$$\xi' - \gamma\eta + \beta\zeta = 0,$$
$$\eta' - \alpha\zeta + \gamma\xi = 0,$$
$$\zeta' - \beta\xi + \alpha\eta = 0,$$

where $\xi, \eta, \zeta$ are direction cosines; $\alpha, \beta, \gamma$ are axial components of an angular velocity $\sigma$; and the accent denotes differentiation with respect to the time. The paper as presented deals only with the case of constant values of $\alpha, \beta, \gamma$. For this case one form of the general solution is the following:

$$\xi = \alpha r_0 + r_1(\alpha\gamma - \beta i\sigma)e^{+i\sigma t} + r_2(\alpha\gamma + \beta i\sigma)e^{-i\sigma t},$$
$$\eta = \beta r_0 + r_1(\beta\gamma + \alpha i\sigma)e^{+i\sigma t} + r_2(\beta\gamma - \alpha i\sigma)e^{-i\sigma t},$$
$$\zeta = \gamma r_0 + r_1(\gamma\gamma - \sigma^2\ )e^{+i\sigma t} + r_2(\gamma\gamma - \sigma^2\ )e^{-i\sigma t}.$$

Excluding the imaginary parts, these equations give

$$\xi = \alpha r_0 + \alpha\gamma r\cos\sigma t + \beta\sigma r\sin\sigma t,$$
$$\eta = \beta r_0 + \beta\gamma r\cos\sigma t - \alpha\sigma r\sin\sigma t,$$
$$\zeta = \gamma r_0 + (\gamma^2 - \sigma)^2 r\cos\sigma t,$$

where $r = r_1 + r_2$.

The differential equations also give

$$\alpha\xi' + \beta\eta' + \gamma\zeta' = 0,$$

whence

$$\alpha\xi + \beta\eta + \gamma\zeta = \sigma\cos\psi,$$

where $\psi$ is the constant angle between the axis of $\sigma$ and the axis to which $\xi, \eta, \zeta$ refer. Hence, since

$$\xi^2 + \eta^2 + \zeta^2 = 1,$$

the constants $r_0$ and $r$ may be expressed in terms of $\psi$ alone by

$$r_0 = \frac{\cos\psi}{\sigma}, \qquad r = \frac{\sin\psi}{\sigma\sqrt{\alpha^2 + \beta^2}},$$

Professor Merriman's paper will be published in the *Journal of the United States Artillery*. The question of the comparative efficiency of the methods of independent and parallel fire from mortar batteries has recently excited great interest in military circles. The case of independent

fire is that where each mortar of the battery is aimed with the intention of hitting the center of a target floating on the water. If $R_1$ and $r_1$ are the probable errors of shot dispersion in range and in azimuth, and $R_2$ and $r_2$ are the probable errors of aim in range and azimuth, it is shown that $R = \sqrt{R_1^2 + R_2^2}$ and $r = \sqrt{r_1^2 + r_2^2}$ are the resultant probable errors by which computations on the probable number of hits are to be made. If $2A$ be the length of the rectangular target in range and $2a$ its width in azimuth, and if $\rho$ denote the constant $0.4769\cdots$, the expression

$$N\left(\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\rho A/R} e^{-t^2}\,dt\right)\left(\frac{2}{\pi^{\frac{1}{2}}}\int_0^{\rho a/r} e^{-t^2}\,dt\right)$$

gives the probable number of hits when $N$ shots are fired. The case of parallel fire where each mortar is aimed parallel to an imaginary mortar at the center of the battery is then discussed. For the rectangular battery whose length is $2B$ parallel to the plane of fire and whose width is $2b$, it is shown that the expression for the probable number of hits from a volley of $N$ shots is

$$N\left(\frac{1}{\pi^{\frac{1}{2}}}\int_{\rho u_1}^{\rho u_2} e^{-t^2}\,dt\right)\left(\frac{1}{\pi^{\frac{1}{2}}}\int_{\rho v_1}^{\rho v_2} e^{-t^2}\,dt\right)$$

where the limits have the values $u_1 = (B - A)/R$, $u_2 = (B + A)/R$, $v_1 = (b - a)/r$, and $v_2 = (b + a)/r$. The conclusion follows that the probable number of hits under independent fire is greater than that under parallel fire. In the long run the advantage of the independent method appears to be about twenty-five per cent.

Professor Brown's "Note on the steering of an eight-oared boat" is an extract from the author's review of Lamb's Hydrodynamics, published in the November number of the BULLETIN (see pp. 77, 78).

Professor Chessin's paper is published in full in this number of the BULLETIN. Dr. Lovett's paper appeared in the November number.

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# NOTE ON HYPERELLIPTIC INTEGRALS.

BY PROFESSOR ALEXANDER S. CHESSIN.

(Read before the American Mathematical Society at the Meeting of October 30, 1897.)

LET $X_r$ denote a polynomial in $x$ of degree $r$; $P_m(x)$, $Q_n(x)$, ... polynomials in $x$ of degrees $m$, $n$, .... We know that the integration of

$$\int f(x, \sqrt{X_r})\, dx,$$

where $f(x, \sqrt{X_r})$ is a rational function of $x$ and $\sqrt{X_r}$, is reduced to the integration of

$$\int \frac{R(x)dx}{\sqrt{X_r}} \tag{1}$$

where $R(x)$ is a rational function of $x$. This note is intended to give a practical rule for the integration of (1).

Let

$$R(x) = \frac{P_m(x)}{\prod_{k=1}^{k=s}(x-a_k)^{n_k}} \tag{2}$$

We may assume that $P_m(x)$ has no factor $x - a_k$, otherwise the common factors may be cancelled. We also assume that all the factors of $X_r$ are simple, for double factors could be taken outside the radical.

Suppose first that none of the $a_k$ are roots of $X_r = 0$. Then we have the equality

$$\int \frac{P_m(x)dx}{\prod_{k=1}^{k=s}(x-a_k)^{n_k}\sqrt{X_r}} = \frac{Q_p(x)\sqrt{X_r}}{\prod_{k=1}^{k=s}(x-a_k)^{n-1}} + \sum_{k=0}^{k=r-2}\lambda_k \int \frac{x^k dx}{\sqrt{X_r}} + \sum_{k=1}^{k=s}\mu_k \int \frac{dx}{(x-a_k)\sqrt{X_r}} \tag{3}$$

where

$$(4)\quad \begin{cases} p = m - s - r + 1 & \text{if} \quad m > \sum_{k=1}^{k=s} n_k + r - 2 \\ p = \Sigma n_k - s - 1 & \text{if} \quad m \leqq \sum_{k=1}^{k=s} n_k + r - 2 \end{cases}$$

In fact after differentiating formula (3) and multiplying the result by $\sqrt{X_r}\prod_{k=1}^{k=s}(x-a_k)^{n_k}$ we have a polynomial of the degree $m$ in the left hand side, and a polynomial of the degree $p+s+r-1$

or $$\sum_{k=1}^{k=s} n_k + r - 2$$

on the right hand side according as

$$m > \sum_{k=1}^{k=s} n_k + r - 2$$

or $$m \leqq \sum_{k=1}^{k=s} n_k + r - 2$$

In the first case by taking for $p$ the value $m-s-r+1$ we obtain a polynomial of degree $m$ on the right hand side; in the second case it will be a polynomial of degree $\leqq m$. In either case we have as many equations to find the indeterminate coefficients of $Q_p(x)$ and the $\lambda_k$ and $\mu_k$ as there are coefficients, namely $m+1$ in the first case and $\sum_{k=1}^{k=s} n_k + r - 1$ in the second.

Suppose now that $a_1, a_2, \cdots, a_\rho$ are roots of $X_r = 0$. Then we have the equality

$$(5)\quad \int \frac{P_m(x)\,dx}{\prod_{k=1}^{k=s}(x-a_k)^{n_k}\sqrt{X_r}} = \frac{Q_q(x)\sqrt{X_r}}{(x-a_1)\cdots(x-a_\rho)\prod_{k=1}^{k=s}x-a_k)^{n_k-1}} + \sum_{k=0}^{k=r-2}\lambda_k\int\frac{x^k dx}{\sqrt{X_r}} + \sum_{k=\rho+1}^{k=s}\mu_k\int\frac{dx}{(x-a_k)\sqrt{X_r}}$$

where

$$(6)\quad \begin{cases} q = m - s - r + \rho + 1 \text{ if } m > \sum_{k=1}^{k=s} n_k + r - 2 \\ q = \sum_{k=1}^{k=s} n_k - s + \rho - 1 \text{ if } m \leqq \sum_{k=1}^{k=s} n_k + r - 1 \end{cases}$$

In fact after differentiating formula (5) and multiplying the result by $\sqrt{X_r}\prod_{k=1}^{k=s}(x-a_k)^{n_k}$ we obtain a polynomial of

degree $m$ on the left hand side, and a polynomial of the degree $q+s+r-\rho-1$ or $\sum_{k=1}^{k=s} n_k + r - 2$ on the right hand side according as $\sum_{k=1}^{k=s} n_k + r - 2 < m$ or $\geqq m$. In the first case by taking for $q$ the value $m-s-r+\rho+1$ we obtain a polynominal of degree $m$ on the right hand side; in the second case the degree will be $\geqq m$. In either case the number of equations to determine the coefficients of $Q_q(x)$ and the $\lambda_k$ and $\mu_k$ is equal to the number of these coefficients, namely $m+1$ in the first case and $\sum_{k=1}^{k=s} n_k + r - 1$ in the second.

That formula (3) does not hold in general when $X_r(a_k)=0$ can be easily seen by substituting for $x$ the value $a_k$ in the result of the differentiation of (3). In fact it will be found that unless $n_k=1$ formula (3) involves the equality $P_m(a_k)=0$ which is contrary to our assumption that $P_m(x)$ has no factor $x-a_k$. *But formula* (3) *still holds if* $a_k$ *is a root of* $X_r=0$ *provided* $n_k=1$. And indeed, in this case formulas (3) and (5) can be brought to the same form if we remember that the integral

$$\int \frac{dx}{(x-a_k)\sqrt{X_r}}$$

is reducible to the integral

$$\int \frac{xdx}{\sqrt{X_r}}$$

when $a_k$ is a root of $X_r=0$.

*Remark.* In order to determine the algebraic part of formulas (3) and (5) it is not necessary to break up the denominator of $R(x)$ into factors. Formula (3) may be written as follows:

$$\int \frac{R(x)dx}{\sqrt{X_r}} = \frac{Q_p(x)\sqrt{X_r}}{\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}} + \sum_{k=0}^{k=r-2} \lambda_k \int \frac{x^k dx}{\sqrt{X_r}} + \int \frac{T_{s-1}(x)dx}{\prod_{k=1}^{k=s}(x-a_k)\sqrt{X_r}}$$

and to find $\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}$ and $\prod_{k=1}^{k=s}(x-a_k)$ we only need to find the greatest common measure of the denominator of

$R(x)$ and its derivative. As to formula (5) it may be written thus

$$\int \frac{R(x)dx}{\sqrt{X_r}} = \frac{Q_t(x)\sqrt{X_r}}{\prod_{k=1}^{k=\rho}(x-a_k)\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\sum_{k=0}^{k=r-2}\lambda_k\int\frac{x^k dx}{\sqrt{X_r}}+\int\frac{T_{s-\rho-1}(x)dx}{\prod_{k=\rho+1}^{k=s}(x-a_k)\sqrt{X_r}}$$

The function $\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}$ is found as just explained, while the functions $\prod_{k=1}^{k=\rho}(x-a_k)$ and $\prod_{k=\rho+1}^{k=s}(x-a_k)$ are found by determining the greatest common measure of the functions $\prod_{k=1}^{k=s}(x-a_k)$ and $X_r$.

The results of this note applied to the case of $r=2$ give very useful formulas, namely

$$(7)\qquad \int\frac{P_m(x)dx}{\prod_{k=1}^{k=s}(x-a_k)^{n_k}\sqrt{X_2}}=\frac{Q_{n-s-1}(x)\sqrt{X_2}}{\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\lambda\int\frac{dx}{\sqrt{X_2}}+\sum_{k=1}^{k=s}\lambda_k\int\frac{dx}{(x-a_k)\sqrt{X_2}}$$

if $X_2(a_k)\neq 0$ $(k\equiv 1, 2, \cdots s)$, $n$ being the greatest of the numbers $m$ and $\sum_{k=1}^{k=s} n_k$; and

$$(8)\quad \int\frac{P_m(x)dx}{\prod_{k=1}^{k=s}(x-a_k)\sqrt{X_2}}=\frac{Q_{n-s+\rho-1}(x)\sqrt{X_2}}{(x-a_1)\ (x-a_\rho)\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\lambda\int\frac{dx}{\sqrt{X_2}}+\sum_{k=\rho+1}^{k=s}\mu_k\int\frac{dx}{(x-a_k)\sqrt{X_2}}$$

if $a_1$, $a_\rho$ are roots of $X_2=0$.

Formula (7) holds also in the case where $a_k$ is a root of $X_2=0$ provided $n_k=1$.

It will be noticed that the method of reduction here used does not require the degree of the numerator of $R(x)$ to be less than that of the denominator.

JOHNS HOPKINS UNIVERSITY,
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# CERTAIN CLASSES OF POINT TRANSFORMATIONS IN THE PLANE.

BY DR. EDGAR ODELL LOVETT.

(Read before the American Mathematical Society at the Meeting of May 29, 1897.)

A POINT transformation is an operation by which a point is carried into the position of some point. As far as the general definition is concerned the path described by the point and the time consumed in the change of position are immaterial, accordingly the coördinates of the final position of the point are functions only of the coördinates of its initial position, and a point transformation of the $xy$-plane into itself is represented analytically by two equations of the form.

$$x_1 = X(x, y), \quad y_1 = Y(x, y), \tag{1}$$

where the functions $X$ and $Y$ are independent analytic functions in the Weierstrassian sense.

By such a transformation point is transformed into point, lineal element* into lineal element, curve into curve, intersecting curves into intersecting curves, curves in contact into curves in contact. By imposing geometrical conditions on the transformation, there result corresponding analytical conditions for the determination of the forms of the functions $X$ and $Y$ and thus particular categories of point transformations arise.

For example, if the transformation (1) is to change straight line into straight line, or in other words, to leave the ordinary differential equation of the second order

$$y'' = 0$$

invariant, the functions $X$ and $Y$ are found to have the forms

$$X \equiv \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad Y \equiv \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}, \tag{2}$$

which define the *general projective transformation* of the $xy$-plane. If, further, the point transformation is to transform parabola into parabola, or what amounts to the same

* The term lineal element is here used in the sense introduced by Lie, namely, to designate the ensemble of a point and a straight line through the point.

thing, to leave invariant the ordinary differential equation of the fourth order

$$5y'''^2 - 3y''y^{iv} = 0,$$

the functions $X$ and $Y$ have the forms

$$X \equiv a_1x + b_1y + c_1, \quad Y \equiv a_2x + b_2y + c_2, \tag{3}$$

which define the so-called *general linear transformation* which leaves the line at infinity invariant. If the point transformation is to leave the areas of all figures in the plane invariant, $X$ and $Y$ have the forms (3) with the additional condition that the determinant

$$\Delta \equiv a_1b_2 - a_2b_1 = 1. \tag{4}$$

If the circular points at infinity are to be invariant by the transformation, $X$ and $Y$ are of the form

$$X \equiv \rho(x\cos\alpha - y\sin\alpha + a), \quad Y \equiv \rho(x\sin\alpha + y\cos\alpha + b); \tag{5}$$

arbitrary $\rho$ gives a so-called *similitudinous transformation* which preserves the forms of figures; $\rho$ equal to unity gives a *Euclidian motion* in the plane. If the product of the radii vectores of the original point and the transformed point is to be constant, say unity, we have the *transformation by inversion* whose $X$ and $Y$ are defined by the equations

$$X \equiv \frac{x}{x^2+y^2}, \quad Y \equiv \frac{y}{x^2+y^2}. \tag{6}$$

Examples might be multiplied further, but it is not to the purpose here. It may be added, however, that some of the most interesting cases are those where $X$ and $Y$ are transcendental functions, notably the *logarithmic* and *exponential* transformations.

It is proposed in this note to determine the forms of $X$ and $Y$ and present a few of the properties of the point transformations of the $xy$-plane respectively defined by the following characteristic properties: 1° Cartesian subtangent of the transformed curve is to be $\frac{m}{n}$ times the Cartesian subtangent of the original curve; 2° Cartesian subnormal $\frac{m}{n}$ times Cartesian subnormal; 3° Cartesian subtangent $\frac{m}{n}$ Cartesian

subnormal; 4° Cartesian subnormal $\frac{m}{n}$ Cartesian subtangent; 5° polar subtangent $\frac{m}{n}$ polar subtangent; 6° polar subnormal $\frac{m}{n}$ polar subnormal; 7° polar subtangent $\frac{m}{n}$ polar subnormal; 8° polar subnormal $\frac{m}{n}$ polar subtangent.

1°. The point transformations of this category are to be of such a nature that point is changed into point and curve transformed into curve in such a manner that the Cartesian subtangent of the transformed curve is equal to $\frac{m}{n}$ times the Cartesian subtangent of the original curve.

The Cartesian subtangent of a curve $f(x,y) = 0$ at a point $(x,y)$ is

$$y\frac{dx}{dy}, \quad \text{or } \frac{y}{y'}; \tag{7}$$

then the defining property of the point transformation sought gives the analytical condition

$$y_1\frac{dx_1}{dy_1} \equiv \frac{m}{n}\, y\frac{dx}{dy}. \tag{8}$$

This condition (8) is now to be turned to account to find the forms of $X$ and $Y$ in equation (1). Substituting in (8) the values of $x_1$ and $y_1$ from (1) there results

$$Y\frac{X_x + X_y y'}{Y_x + Y_y y'} \equiv \frac{m}{n}\frac{y}{y'},$$

or

$$nX_y y'^2 + (nYX_x - myY_y)\, y' - myY_1 \equiv 0. \tag{9}$$

This last equation must be identically true for all values of $y'$, hence equating to zero the coefficients of the several powers of $y'$

$$X_y \equiv 0, \quad Y_x \equiv 0, \quad nYX_x - myY_y \equiv 0 \tag{10}$$

hence

$$X = X(x), \quad Y = Y(y), \quad X_x = m, \quad \frac{Y_y}{Y} = \frac{n}{y}\,* \tag{11}$$

* The last identity (10) breaks up into these two parts since $X$ and $Y$ are independent functions by hypothesis.

and finally

$$X = mx + a, \quad Y = by^n. \tag{12}$$

Then the transformation

$$x_1 = mx + a, \quad y_1 = by^n \tag{13}$$

is the most general point transformation which changes a curve $c$ into a curve $\gamma$ in such a manner that the subtangent of the point $(x, y)$ of $c$ is the $\frac{n}{m}$ th part of the subtangent of the corresponding point $(x_1, y_1)$ of the transformed curve $\gamma$.

2°. The point transformation is to be found which changes curve into curve in such a manner that the Cartesian subnormal of the transformed curve is $\frac{m}{n}$ times the Cartesian subnormal of the original curve. The value of the Cartesian subnormal is

$$y\frac{dy}{dx}. \tag{14}$$

By virtue of the transformation there exists an identity of the form

$$y_1 y_1' \equiv \frac{m}{n} yy'. \tag{15}$$

Whence

$$myX_x y'^2 + (nYY_y - myX_x)y' - nYY_x \equiv 0. \tag{16}$$

This identity must obtain for all values of $y'$, hence as in the preceding case

$$X = X(x), \quad Y = Y(y), \quad X_x = n, \quad YY_y = my, \tag{17}$$

therefore

$$x_1 \equiv X = nx + a, \quad y_1 \equiv Y = \sqrt{my^2 + b} \tag{18}$$

is the most general point transformation of the characteristic property 2°.

3°. By this third class of point transformations the Cartesian subtangent of the transformed curve is to be equal to the Cartesian subnormal of the original curve multiplied by the ratio $\frac{m}{n}$. This geometrical property expressed analytically becomes

$$\frac{y_1}{y_1'} \equiv \frac{m}{n} yy'. \tag{19}$$

Whence

$$myY_y y'^2 + (myY_x - nYX_y)y' - nYX_x \equiv 0; \tag{20}$$

hence

$$X = X(y), \quad Y = Y(x), \quad X_y = my, \quad Y_x = nY; \tag{21}$$

accordingly

$$x_1 = \frac{m}{2} y^2 + a, \quad y_1 = be^{nx} \tag{22}$$

are the equations of the most general point transformation possessing the assigned property.

4°. The defining property of this category of point transformation is that the Cartesian subnormal of the transformed curve shall be to the Cartesian subtangent of the original curve in the ratio $m$ to $n$; hence the condition

$$y_1 y_1' \equiv \frac{m}{n} \frac{y}{y'} \tag{23}$$

which gives

$$nYY_y y'^2 + (nYY_x - myX_y)y' - myX_x \equiv 0. \tag{24}$$

Therefore

$$X = X(x), \quad Y = Y(y), \quad yX_y = n, \quad YY_x \equiv m; \tag{25}$$

and finally

$$x_1 = n \log y + a, \quad y_1 = \sqrt{2mx + b} \tag{26}$$

are the equations of the transformation sought.

5°. The next four transformations are more readily studied in polar coördinates. Let the general point transformation of the $(r\theta)$ plane into itself be

$$r_1 = R(r, \theta), \quad \theta_1 = \theta(r, \theta) \tag{27}$$

The transformation 5° is subject to the limitation that the polar subtangent of the transformed curve be to the polar subtangent of the original curve in the ratio $m$ to $n$; this geometrical condition gives the following analytical condition for determining the forms of the functions $R$ and $\theta$,

$$r_1^2 \frac{d\theta_1}{dr_1} \equiv \frac{m}{n} r^2 \frac{d\theta}{dr}, \tag{28}$$

or

$$mr^2 R_\theta \theta'^2 + (mr^2 R_r - nR^2\theta_\theta)\theta' - nR^2\theta_r \equiv 0; \tag{29}$$

hence

$$R = R(r), \quad \theta = \theta(\theta), \quad r^2 R_r = nR^2, \quad \theta_\theta = m; \tag{30}$$

therefore

$$r_1 \quad R \equiv \frac{r}{ar+n} \qquad \theta_1 \equiv \theta = m\theta + \beta \tag{31}$$

are the equations of the most general point transformation possessing the above-named property.

Similarly for the other cases we find:

6° Polar subnormal $\frac{m}{n}$ times polar subnormal,

if $$nR_\theta \theta'^2 + (nR_r - m\theta_\theta)\theta' - m\theta_r \equiv 0; \tag{32}$$

whence $$r_1 = mr + \alpha, \quad \theta_1 = n\theta + \beta. \tag{33}$$

7°. Polar subtangent $\frac{m}{n}$ polar subnormal,

if $$nR^2\theta_\theta \theta'^2 + (nR^2\theta_r - mR_\theta)\theta' - mR_r \equiv 0; \tag{34}$$

whence $$r_1 = \frac{1}{\alpha - n\theta}, \quad \theta_1 = mr + \beta. \tag{35}$$

8°. Polar subnormal $\frac{m}{n}$ solar subtangent,

if $$mr^2\theta_\theta \theta'^2 + (mr^2\theta_r - nR_\theta)\theta' - nR_r \equiv 0; \tag{36}$$

whence $$r_1 = m\theta + \alpha, \quad \theta_1 = \beta - \frac{n}{r}.$$

9. A family of transformations is said to form a Lie group of transformations when the product of any two transformations of the family is equivalent to a transformation belonging to the family. By the product of any number of transformations is meant the transformation equivalent to their successive application. It is to be observed that these transformation products do not obey exactly the same laws as ordinary algebraic products; they always follow the associative law but do not of necessity obey the commutative law.

Consider the family of transformations 1°. Let $S$ be the transformation of the family which changes the point $(x, y)$ into the point $(x_1, y_1)$ given by,

$$x_1 = mx + a, \quad y_1 = by^n; \qquad S \qquad (37)$$

let $T$ be the transformation of the same family which transforms the point $(x_1, y_1)$ into the point $x_2, y_2)$ given by the equations, say

$$x_2 = m_1 x_1 + a_1, \quad y_2 = b_1 y_1^{n_1}; \qquad T \qquad (38)$$

the product $ST$, that is, the transformation which changes the original point $(x, y)$ directly into the point $(x_2, y_2)$ is obtained by eliminating $x_1$ and $y_1$ from the equations (37) and (38); this elimination yields

$$\left.\begin{aligned} x_2 &= mm_1 x + m_1 a + a_1 = m_2 x + a_2, \\ y_2 &= b^{n_1} b_1 y^{nn_1} = b_2 y^{n_2}. \end{aligned}\right\}; \; ST \equiv V \qquad (39)$$

the equations (39) are of the same form as the equations (37) and (38) *i. e.*, the transformation $V$ equivalent to the successive application of the transformations $S$ and $T$ of the family 1° has the same form as $S$ and $T$ and hence belongs to the family 1°; this remarkable property is compressed into the statement that the family of point transformations 1° constitutes a continuous group of transformations.

Similarly it may be shown that the transformations 2° form a continuous group of transformations.

On the contrary, the transformations 3° do not constitute a group, as the following consideration shows. Let $P$ be the transformation of the family 3° which changes the point $(x, y)$ into the point $(x_1, y_1)$ given by

$$x_1 = \frac{m}{2} y^2 + a, \quad y_1 = be^{nx}; \qquad P \qquad (40)$$

the point $(x_1, y_1)$ is changed by a transformation $Q$ of this same family into the point $(x_2, y_2)$ for which, say

$$x_2 = \frac{m_1}{2} y_1^2 + a_1, \quad y_2 = b_1 e^{n_1 x}; \qquad Q \qquad (41)$$

the transformation $W$ equivalent to the successive application of $P$ and $Q$ is found by eliminating $x_1$ and $y_1$ from the equations (40) and (41). This elimination gives the trans-

formation which changes $(x, y)$ directly into $(x_2, y_2)$ and yields as the equations of the transformation

$$x_2 = \frac{m_1}{2} b^2 e^{2nx} + a_1, \; y_2 = b_1 e^{n_1 \frac{m}{2} e^{2} + n_1 a}; PQ \equiv W \qquad (42)$$

but the equations (42) are not of the form (40), hence $W$, the product of $P$ and $Q$, does not belong to the family containing $P$ and $Q$, and the family of transformations 3° does not form a continuous group of transformations.

In the same manner it appears that the transformations 4° do not form a group.

It may also be verified readily that the families 5° and 6° possess the group property; and that the families 7° and 8° are not continuous groups.

10. An *infinitesimal* point transformation is one by which a point suffers an infinitesimal change of position. It differs from the identical transformation by an infinitesimal. If, by virtue of an infinitesimal point transformation, $x$ and $y$ receive the increments respectively

$$\delta x = \xi(x,y)\delta t, \quad \delta y = \eta(x,y)\delta t,$$

the infinitesimal transformation is represented symbolically by

$$Vf \equiv \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y}.*$$

The equations of the group 1° give the identical transformation

$$x_1 = x, \quad y_1 = y$$

for the system of values

$$a = 0, \quad b = 1, \quad m = 1, \quad n = 1;$$

hence the system

$$a = \delta a, \quad b = 1 + \delta b, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

yields a transformation of the form 1° differing from the identical transformation by an infinitesimal, and the transformation corresponding to this system is an infinitesimal transformation of the group 1°; accordingly

$$x_1 = (1 + \delta m)x + \delta a,$$

$$y_1 = (1 + \delta b)y^{1+\delta n} = y(1 + \delta b)(1 + \frac{\partial n \log y}{1} + \cdots)$$

* For these details consult any one of Lie's published works.

are the equations of the infinitesimal transformation of the group 1°. Whence

$$x_1 - x \equiv \delta x = x\delta m + \delta a, \quad y_1 - y \equiv \delta y = y(\delta b + \delta n \log y)$$

to terms of the second order; therefore, if we make an obvious change in the designation of the infinitesimal constants $\delta a, \delta b, \cdots$, we have

$$\delta x = \xi(x, y)\delta t = (ax + \beta)\delta t, \quad \delta y = \eta(x, y) = y(\lambda + \mu \log y)\, \delta t$$

and the symbol of the infinitesimal transformation of our group 1° is

$$Vf \equiv (ax + \beta)\frac{\partial f}{\partial x} + y(\lambda + \mu \log y)\frac{\partial f}{\partial y}.$$

The system of values

$$a = 0, \quad b = 0, \quad m = 1, \quad n = 1$$

makes 2° an identical transformation, hence the system of values

$$a = \delta a, \quad b = \delta b, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

determines an infinitesimal transformation of the group 2°, namely

$$x_1 = (1 + \delta n)x + \delta a, \quad y_1 = \sqrt{(1 + \delta m)y^2 + \delta b}\,;$$

whence

$$x_1 - x \equiv \delta x = x\delta n + \delta a, \quad y_1 - y \equiv \delta y = \frac{\delta m}{2}y + \frac{\delta b}{y}$$

by expanding $y_1$ by the binomial formula and by neglecting terms of higher order. Then putting

$$\delta n = \lambda \delta t, \quad \delta a = \mu \delta t, \quad \delta m = \rho \delta t, \quad \delta b = \sigma \delta t,$$

the infinitesimal transformation of the group 2° has the symbolic form

$$Vf \equiv (\lambda x + \mu)\frac{\partial f}{\partial x} + (\rho y + \sigma y^{-1})\frac{\partial f}{\partial y}.$$

Since the families 3° and 4° do not possess the identical

transformation they have no infinitesimal transformations.*

11. We have already found that the family of transformations 5° is a group, hence we shall find an infinitesimal transformation belonging to the family. In fact the following values of the arbitrary constants

$$a = 0, \quad \beta = 0, \quad m = 1, \quad n = 1$$

reduce the transformation 5° to the identical transformation

$$r_1 = r, \quad \theta_1 = \theta.$$

Then the system of values

$$a = \delta a, \quad \beta = \delta\beta, \quad m = 1 + \delta m, \quad n = 1 + \delta n$$

determines the transformation of the group which differs from the identical transformation by an infinitesimal, that is, the infinitesimal transformation of the group, namely,

$$r_1 = \frac{r}{r\delta\beta + 1 + \delta n}, \qquad \theta_1 = (1+\delta m)\,\theta + \delta a$$

or to terms of the second order

$$r_1 = r\,(1 - \delta n - r\delta\beta), \quad \theta_1 = (1 + \delta m)\theta + \delta a;$$

whence

$$\delta r \equiv r_1 - r = -r(x\delta\beta + \delta n), \quad \delta\theta \equiv \theta_1 - \theta = \theta\delta m + \delta a;$$

or putting

$$\delta\beta = \lambda\delta t, \quad \delta n = \mu\delta t, \quad \delta m = \nu\delta t, \quad \delta a = \rho\delta t,$$

the symbol of the infinitesimal transformation of the group 5° is

$$Vf \equiv \xi(r,\theta)\frac{\partial f}{\partial r} + \eta(r,\theta)\frac{\partial f}{\partial\theta} \equiv (\nu\theta + \rho)\frac{\partial f}{\partial\theta} - r(\lambda r + \mu)\frac{\partial f}{\partial r}.$$

In the same manner we find the infinitesimal transformation of the group 6° to have the symbol

$$Vf \equiv (fr + g)\frac{\partial f}{\partial r} + (h\theta + k)\frac{\partial f}{\partial\theta}.$$

The families 7° and 8° do not contain the identical transformation, hence they do not have infinitesimal transformations.*

* It is to be observed that it is not for the reason that these families are not groups that they do not contain infinitesimal transformations. For example, the family of $\infty^1$ transformations $x_1 = xt$, $y_1 = y + t - 1$, obviously does not form a group, and yet the family contains the identical transformation, as is seen by putting $t$ equal to unity, and the infinitesimal transformation $x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$ found by putting $1 + \delta t$ for the parameter.

12. Apart from their properties as transformations, the above transformations are of interest because of certain applications to plane curves, notably to spirals which it is hoped to bring out in a subsequent note.

Since finishing this note the writer finds that the *finite* forms of the transformations discussed were given by Laisant in the *Nouvelles Annales de Mathématiques*, 2d series, vol. 7 (1868), p. 318, in the solution of a problem proposed by Haton de la Goupillière, *Nouvelles Annales*, vol. 6 (1867), problem No. 803. The wide divergence between the properties and the points of view of the present note and the solution referred to seem to warrant its presentation to the Society. The above-mentioned volumes of the *Nouvelles Annales* are to be had in the Library of Congress.

BALTIMORE,
14 *April*, 1897.

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## CONTINUOUS GROUPS OF CIRCULAR TRANSFORMATIONS.*

BY PROFESSOR H. B. NEWSON.

(Read before the American Mathematical Society, at the Meeting of April 24, 1897.)

THE object of this paper is to present the outlines of a fairly complete theory of the continuous groups of linear fractional transformations of one variable. The method employed is quite different from the methods of Lie. Lie's classic theory is based upon the infinitesimal transformation; I shall make but little use of the infinitesimal transformation, but shall develop the subject from the consideration of the essential parameters of the transformation. The complex plane is chosen because it beautifully illustrates the methods. I have put together some old and some new facts and have sought to build up a general theory.

* Several terms have been proposed to designate the linear fractional transformations of the complex plane. Möbius introduced the term "Kreisverwandtschaft." Mathews' Theory of Numbers, page 107, translates this as "Möbius' Circular Relation." Professor Cole, in *Annals of Mathematics*, vol. 5, page 137, refers to "Orthomorphic Transformation," following Cayley; this seems too general for the special case here considered, since it is applicable to all conformal transformations. Darboux, in his Theorie des Surfaces, vol. 1, page 162, uses "transformation circulaire." It seems to me that "Circular Transformation" is the best yet proposed, for the fundamental property is expressed in the name.

A circular transformation $T$ of the complex plane is represented by

$$z_1 = \frac{az + b}{cz + d}.$$

Regarding the straight lines of the plane as circles through the one point at infinity, the fundamental property of this transformation is that it transforms circles into circles. It interchanges among themselves the circles of the plane, but leaves unchanged or invariant the configuration composed of the $\infty^3$ circles of the plane. (See Forsyth's Theory of Functions, pages 512–524.)

Since any two circular transformations $T$ and $T_1$ each leave invariant this configuration of all circles of the plane, their product, *i. e.*, the transformation which is equivalent to the successive application of the two, must likewise leave the same configuration invariant and hence be a circular transformation.

This conclusion may be verified analytically by eliminating $z_1$ from two circular transformations $T$ and $T_1$ as follows:

$$(1) \qquad z_1 = \frac{az + b}{cz + d}, \text{ and } z_2 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}.$$

The product of $T$ and $T_1$ is $T_2$ given by

$$(2) \qquad z_2 = \frac{(a_1 a + b_1 c)z + (a_1 b + b_1 d)}{(c_1 a + d_1 c)z + (c_1 b + d_1 d)}.$$

This has the same form as $T$ and therefore is a circular transformation.

But this is just what is known in modern mathematical language as the *group property*. Hence all circular transformations of the complex plane form a group.

If the coefficients $a, b, c, d$ in (1) be made to vary continuously, all the resulting transformations belong to the above group; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in (1). Such a group is called a *continuous* group. The question of continuity will be more fully discussed later.

The circular transformation $T$ can usually be brought to the *normal form**

* Mathews: Theory of Numbers, p. 105.

$$\frac{z_1 - m}{z_1 - n} = k\frac{z - m}{z - n} \tag{3}$$

Where $m$ and $n$ are the roots of the quadratic equation $cz^2 + (d - a)z - b = 0$, and

$$k = \frac{(a + d - \sqrt{(a + d)^2 - 4(ad - bc)})^2}{4(ad - bc)}.$$

$m$, $n$ and $k$ are called the *essential parameters* of the transformation, $m$ and $n$ are the two invariant points and $k$ is the multiplier of the transformation. Since

$$k = \frac{z_1 - m}{z_1 - n} : \frac{z - m}{z - n},$$

we infer that $k$ is the anharmonic ratio of the two invariant points and any pair of corresponding points in the transformation. Since $m$, $n$ and $k$ are complex quantities of the form $a + ib$, it follows that $T$ involves six real variable parameters.

When the invariant points $m$ and $n$ coincide, $T$ can no longer be brought to the above normal form but is then reducible to a *second normal form*

$$\frac{1}{z_1 - m'} = \frac{1}{z - m'} + a. \tag{4}$$

The condition for coincident invariant points is $(a + d)^2 = 4(ad - bc)$.

$$m' = \frac{a - d}{2c} \text{ and } a = \frac{2c}{a + d}.$$

$a$ is a constant whose properties are to be determined. When the condition for two coincident invariant points is substituted in the equation for $k$, we get $k = 1$. Hence the characteristic anharmonic ratio of a transformation $T'$ of this kind is unity. Every circular transformation can be brought to the one or the other of these normal forms.

Let us consider two transformations $T$ and $T_1$ which have no invariant point in common. Their equations are

$$\frac{z_1 - m}{z_1 - n} = k\frac{z - m}{z - n} \quad \text{and} \quad \frac{z_2 - m_1}{z_2 - n_1} = k_1\frac{z_1 - m_1}{z_1 - n_1}. \tag{3'}$$

Eliminating $z_1$ we have the product $T_2$ in the form

$$\frac{z_2 - m_2}{z_2 - n_2} = k_2\frac{z - m_2}{z - n_2} \tag{3''}$$

where $k_2$, $m_2$, and $n_2$ are given as follows:

$$(5)\quad \begin{cases} \dfrac{k_2+1}{\sqrt{k_2}} = \dfrac{(kk_1-k-k_1+1)}{\sqrt{kk_1}}\lambda + \dfrac{k+k_1}{\sqrt{kk_1}}; \\ m_2 = \dfrac{k_2 D + A}{C(k_2+1)}; \quad n_2 = \dfrac{k_2 A + D}{C(k_2+1)}. \end{cases}$$

$A$, $D$, $C$ and $\lambda$ stand for the following expressions:

$$(6)\quad \begin{cases} A = kk_1n_1(n-m_1) - km_1(n-n_1) - k_1n_1(m-m_1) \\ \qquad + m_1(m-n_1). \\ D = kk_1m(n-m_1) - km(n-n_1) - k_1n(m-m_1) \\ \qquad + n(m-n_1). \\ C = kk_1(n-m_1) - k(n-n_1) - k_1(m-m_1) + (m-n_1). \\ \lambda = \dfrac{n_1-m}{m_1-n_1} : \dfrac{n-m}{m_1-n}; \end{cases}$$

*i. e.*, $\lambda$ is one of the anharmonic ratios of the four invariant points $(mnm_1n_1)$.

The transformation $T_2$ is not independent of the order of the components $T_1$ and $T$; the value of $k_2$ is independent of the order of $T_1$ and $T$ for $\lambda$ is unaltered when $m$ and $n$ are interchanged with $m_1$ and $n_1$ but not so with $m_2$ and $n_2$. Hence the two transformations $T$ and $T_1$ are non-commutative.

The results obtained may be formulated as follows:

THEOREM 1. *All circular transformations of the complex plane form a six-parameter continuous group. The transformations of the group are non-commutative.*

Our task is now to enumerate and discuss all the subgroups of this six-parameter group, to develop their properties and to classify them according to their most characteristic properties.

Lie expounds in "Continuierliche Gruppen," page 113 an axiomatic principle which, for the purposes of this paper, is best stated in the following form: *All point transformations of the plane which leave invariant a certain figure or configuration in the plane have the group property.* The group may be either a continuous or a discontinuous group. A good example of the latter is the group of 18 linear transformations of the plane cubic into itself. We shall make frequent use of this principle in what follows; but in each case the group obtained will be seen to contain one or more continuously varying parameters and is therefore a continuous group.

According to Lie's principle all transformations leaving invariant a single point $m$ form a group. Since there are $\infty^2$ points in the plane, the $\infty^6$ transformations of the six-parameter group $G_6$ are distributed into $\infty^2$ subgroups, one for each point. Accordingly each such subgroup should contain $\infty^4$ transformations and be a four-parameter group $G_4$ or $G_m$. This group leaves invariant not only the point $m$ but also the net of circles through $m$. The circles of the net are interchanged among themselves, but the net as a whole is unaltered.

If we make $m_1 = m$ in (3′), (5), and (6), we find $\lambda = 1$, $k_2 = kk_1$, and $m_2 = m$. If $m$ be a fixed point, the equations (3′) or (3″) contain only two essential variable parameters $k$ and $n$ and, hence, four real variable parameters; thus, it is shown analytically that the group $G_m$ is four-parameter. The fact that $k_2 = kk_1$ is very important as will be seen later. The transformations of the group are non-commutative, for $n_2$ is not independent of the order of $T$ and $T_1$.

THEOREM 2. *All transformations leaving a point m invariant form a four-parameter group. The law of combination of the essential parameters k of this group is given by* $kk_1 = k_2$. *The transformations of the group are non-commutative.*

Again by Lie's principle all transformations leaving invariant two distinct points $m$ and $n$ form a group. Since there are $\infty^4$ such pairs of points, the transformations of the six-parameter group $G_6$ are distributed into $\infty^4$ subgroups each of which contains $\infty^2$ transformations and is a two-parameter group. It is clear that each four-parameter group $G_m$ contains $\infty^2$ of these two-parameter groups $G_2$ or $G_{mn}$, one corresponding to each point of the plane taken with the fixed point $m$. Such a two-parameter group leaves invariant not only the points $m$ and $n$ but also the pencil of circles through these points.

If we make $m_1 = m$ and $n_1 = n$ in (3′), (5) and (6), we get $\lambda = 1$, $k_2 = kk_1$, $m_2 = m$ and $n_2 = n$. Or we may eliminate $z_1$ from (3′) and get (3″) by multiplication and thus get directly that $k_2 = kk_1$. $T$ in this case has only one essential variable parameter $k$ and, hence, only two real variable parameters. Thus again the group $G_{mn}$ is shown to be two-parameter. Since $k_2$ is independent of the order of $k$ and $k_1$, the transformations of this group are commutative.

THEOREM 3. *All transformations leaving invariant two points m and n form a two-parameter group. The law of combination of the essential parameters of this group is expressed by* $k_2 = kk_1$. *The transformations of the group are commutative.*

Thus far we have considered only transformations of the type $T$ with two invariant points. The normal form of $T$

(4) contains only two essential parameters $m'$ and $a$ and hence only four real variable parameters. There are therefore $\infty^4$ such transformations in the plane. Taken together do they form a group? This is readily answered in the negative. The product of two transformations $T'$ and $T_1'$

$$(4') \qquad \frac{1}{z_1-m'}=\frac{1}{z-m'}+a, \text{ and } \frac{1}{z_2-m_1'}=\frac{1}{z_1-m_1'}+a_1,$$

is a transformation of the kind $T$ with two distinct invariant points, as may easily be verified by eliminating $z_1$ from (4').

But if the two transformations $T'$ and $T_1'$ leave invariant the same point $m'$, they do form a group. Making $m_1'=m'$ in (4') and eliminating $z_1$ by addition, we get

$$(4'') \qquad \frac{1}{z_2-m'}=\frac{1}{z-m'}+a_2;$$

where $a_2=a+a_1$. The group is evidently two-parameter and its transformations are commutative. Let it be symbolized by $G_\infty'$ or $G_2'$.

THEOREM 4. *All transformations of the kind $T'$ leaving a single point invariant form a two-parameter group. The law of combination of the essential parameters is expressed by $a_2=a+a_1$. The transformations of the group are commutative.*

The relationships of the groups thus far determined may be symbolized as follows:

$$G_6=\infty^2 G_4=\infty^4 G_2+\infty^2 G_2'. \qquad G_4=\infty^2 G_2+G_2'$$

or

$$G_6=\infty^2 G_4=\infty^2(\infty^2 G_2+G_2').$$

We now go on to examine more closely the two-parameter group $G_{\infty\infty}$ and shall show that the transformations composing it can be distributed into one-parameter subgroups. The essential parameter $k$ of the group $G_{\infty\infty}$ may be written $k=\rho e^{i\theta}$. Here $\rho$ and $\theta$ are independent parameters and may vary independently. If we put $\rho=e^c$, where $c$ is some constant quantity, we have $k=e^{c}\cdot e^{i\theta}=e^{(c+i)\theta}$. Since in the group $G_{\infty\infty}$ $k_2=kk_1$, we have $k_2=e^{(c+i)\theta}\cdot e^{(c_1+i)\theta_1}=e^{c\theta+c_1\theta_1}\cdot e^{i(\theta+\theta^1)}$, where $c$ and $\theta$ are two independent parameters. But if $c=c_1$, we have $k_2=e^{c(\theta+\theta_1)}\cdot e^{i(\theta+\theta_1)}=e^{(c+i)\theta_2}$, where $\theta_2=\theta+\theta_1$. Here we have the conditions for a one-parameter group; $k_2$ is of the same character as $k$ and $k_1$, and there is but one parameter, viz: $\theta$. The effect of the successive transformations of the group upon a point $P$ of the plane is to transform it

into $P_1, P_2, \cdots P_n$ which lie on a curve called by Lie the Bahncurve or path curve. By a transformation $T$ every point $P$ is moved along its path curve. The path curves of this group are the well-known double spirals of Holzmüller* used so extensively by Klein and others. Their properties are so well known that it is unnecessary to develop them here. For a lucid account in English see a paper by Professor F. N. Cole in *Annals of Mathematics*, vol. 5, page 121.

Within the group $G_{mn}$ there is a sub-fold group for every real value of $c$ and a corresponding set of double spiral path curves.

THEOREM 5. *The transformations of the group $G_{mn}$ are distributed into $\infty^1$ one-parameter subgroups. The path curves of these subgroups are double spirals about the invariant points $m$ and $n$.*

The chief properties of one of these one-parameter subgroups are easily deduced from the expression for the parameter, $k = e^{(c+i)\theta}$. When $\theta = 0$, $k = 1$; when the anharmonic ratio of the four points $(mnzz_1) = 1$, $z$ coincides with $z_1$. Every point of the plane is unaltered by such a transformation, which is called the identical transformation of the group. The two transformations corresponding to values of $\theta$ numerically equal but of opposite signs are called inverse transformations. Their product is the identical transformation of the group. When $k = \infty$ or 0, all points of the plane are transformed respectively to the invariant points $m$ or $n$. I have elsewhere called these pseudo-transformations† (*ausgeartete Transformationen*, Lie).

Within the two-parameter group $G_{mn}$ are two one-parameter subgroups of special importance; these are the groups for which $c = 0$ and $c = \infty$ respectively; *i. e.*, for which $|k| = 1$ and for which $k$ is real. In the first case, for which $k = e^{i\theta}$, the path curves reduce to coaxial circles having $m$ and $n$ for vanishing points. All transformations of this one-parameter group are elliptic. In the second case, when $k$ is real, the path curves reduce to a pencil of circles through $m$ and $n$. The transformations of this group are all hyperbolic. The other one-parameter subgroups of $G_{mn}$ are made up chiefly of loxodromic transformations.

THEOREM 6. *Every two-parameter group $G_{mn}$ contains one one-parameter subgroup of elliptic transformations, one one-parameter subgroup of hyperbolic transformations, and $\infty^1$ one-parameter subgroups of loxodromic transformations. For each group of loxodromic transformations $c$ in the formula $k = e^{(c+i)\theta}$ is a constant.*

The continuity of the two-parameter group $G_{mn}$ is based

* See Holzmüller: Isogonale Verwandtschaften, § 19.

† *Kansas University Quarterly*, vol. 5, page 79.

upon the continuity of the complex number system; for there is a transformation of the group corresponding to every value of $k$, which is a complex number. Let the values of $k$ be represented as usual by the points of a complex plane (not to be confused with the plane of our operations). We wish to see how the values of $k$ which give transformations belonging to a one-parameter subgroup are distributed in the plane. We have $k = e^{(c+i)\theta}$ where $c$ is a constant and $\theta$ is variable. The locus of the point $k$ satisfying this equation is a logarithmic spiral about the zero point cutting the axis of real numbers at an angle whose cotangent is $c$. This is a continuous curve from the zero point to the infinity point, and consequently our one-parameter subgroup is a continuous group.

Different values of $c$ give us different spirals each of which corresponds to a one-parameter subgroup of $G_{mn}$. $c$ varies continuously through all real values from $-\infty$ to $+\infty$ so that these spirals lie infinitely close to one another and, as we shall see, cover twice over the entire plane. These spirals all pass through the unit point. For $c = 0$ the corresponding spiral becomes the unit circle; for $c=\infty$ the spiral reduces to the straight line which is the axis of real numbers. The family of spirals for which $c$ is positive fills the entire plane and no two of them intersect except in the unit point. The same is true of the family of spirals for which $c$ is negative. But every spiral of one family intersects an infinite number of times every spiral of the other family. Every point in the plane not on the unit circle or the axis of real numbers lies on two of these spirals; from which we infer that every loxodromic transformation of the group $G_{mn}$ belongs to two distinct one-parameter subgroups. Every hyperbolic transformation in $G_{mn}$ except the involutoric transformation, for which $k = -1$, belongs to three one-parameter subgroups; for two spirals and the axis of reals pass through every point for which $k$ is real. The elliptic transformations in $G_{mn}$ belong only to the elliptic subgroup. The involutoric transformation is common to the hyperbolic and elliptic subgroups. The identical transformation is common to all subgroups; and the two pseudo-transformations are common to all except the elliptic subgroup. Two loxodromic subgroups for which the $c$'s have the same signs have no transformations in common other than the identical and the two pseudo-transformations; while two loxodromic subgroups for which the $c$'s have opposite signs have in common an infinite number of discrete transformations.

THEOREM 7. *Every one-parameter subgroup in $G_{\infty\infty}$ is continuous. Every loxodromic transformation in $G_{\infty\infty}$ belongs to two distinct subgroups. Every hyperbolic transformation in $G_{\infty\infty}$, except the involutoric transformation, belongs to three distinct subgroups.*

This same geometric representation enables us to discuss intuitively the generation of finite transformations by the repetition of an infinitesimal transformation. Every spiral passes through the unit point, and corresponding to the two points on the spiral adjacent to the unit point we have two infinitesimal transformations belonging to a one-parameter group. These are given by $k = e^{+(c+i)\delta\theta}$ and $k = e^{-(c+i)\delta\theta}$. The identical transformation divides the one-parameter group into two portions, each of which contains an infinitesimal transformation. Every finite transformation in each portion of a one-parameter loxodromic group can be generated by the repetition of the corresponding infinitesimal transformation. In the elliptic group, for which the spiral reduces to a circle, every transformation can be generated from either elliptic infinitesimal transformation. In the hyperbolic group, for which the spiral reduces to the axis of real numbers, the transformations for which $k$ is negative can not be generated by the repetition of either of the hyperbolic infinitesimal transformations. Every loxodromic transformation in $G_{\infty\infty}$ can be generated from either of two distinct infinitesimal transformations, for every loxodromic transformation belongs to two distinct subgroups. Every hyperbolic transformation for which $k$ is positive can be generated from three infinitesimal transformations; while every hyperbolic transformation for which $k$ is negative, except the involutoric transformation, can be generated from two distinct loxodromic transformations.

THEOREM 8. *Every hyperbolic transformation in $G_{\infty\infty}$ for which $k$ is positive can be generated from three distinct infinitesimal transformations; every other transformation in $G_{\infty\infty}$ can be generated from two distinct infinitesimal transformations.*

The two-parameter group $G_\infty'$ likewise contains $\infty^1$ one-parameter subgroups. The law of combination of the parameters in this group is $a_2 = a + a_1$, or in another form $r_2 e^{i\theta_2} = re^{i\theta} + r_1 e^{i\theta_1}$. If now we take $\theta_1 = \theta$, this becomes $r_2 e^{i\theta} = (r + r_1) e^{i\theta}$. We have here the conditions for a one-parameter group; $a_2$ is of the same form as $a$ and $a_1$ and contains only one parameter $r$. It is clear that we have a one-parameter group for every value of $\theta$. The effect of successive applications of transformations of one of these one-parameter groups on a point in the plane is to move it

along a path curve. The path curves of one of these groups consists of the system of circles tangent at $m$ to each other and to the line through $m$ which makes with the axis of reals an angle $\theta$. All transformations of the group $G_m'$ are parabolic. For details see the above mentioned paper by Professor Cole.

The properties of one of these one-parameter groups are easily determined. Let $a = re^{i\theta}$; when $r = 0$, we have the identical transformation of the group; the two transformations corresponding to two values of $r$ equal but with opposite signs are inverse transformations. When $r = \infty$, all points of the plane are transformed to $m$ and we have a pseudo-transformation. There are two infinitesimal transformations in the group given by the values $+ dr$ and $- dr$. Each infinitesimal transformation generates its corresponding portion of the group. Two one-parameter subgroups of $G_m'$ have no transformation in common except the identical and the pseudo-transformation; these are common to all subgroups of $G_m'$.

THEOREM 9. *All transformations of the two-parameter group $G_m'$ are parabolic and are distributed into $\infty^1$ one-parameter subgroups. The path curves of a one-parameter subgroup are circles through $m$, touching each other at $m$.*

We have already shown how the four-parameter group $G_m$ breaks up into $\infty^2$ two-parameter groups $G_{mm}$. We shall now show that the transformations of $G_m$ may be distributed into $\infty^1$ three-parameter subgroups. The law of combination of the parameters $k$ within the group $G_m$ is expressed (Theorem 2) by $kk_1 = k_2$. Written in another form this is $e^{(c+i)\theta} \cdot e^{(c_1+i)\theta_1} = e^{c\theta+c_1\theta_1} e^{i(\theta+\theta_1)}$. If $c = c_1$, we have

$$e^{(c+i)\theta} \cdot e^{(c+i)\theta_1} = e^{(c+i)\theta_2} \text{ where } \theta_2 = \theta + \theta_1.$$

Hence, we see that if we chose from each two-parameter group $G_{mm}$ the one-parameter group characterized by a certain value of $c$, the totality of the transformations comprised in these $\infty^2$ one-parameter groups forms a three-parameter group. It is clear at once that there is one such three-parameter group for every value of $c$. Thus, for example, all the elliptic transformations contained in $G_m$ form a three-parameter subgroup. The same is true of all hyperbolic transformations.

THEOREM 10. *The $\infty^3$ transformations having a common invariant point at $m$, and for which the value of $c$ in the formula $k = e^{(c+i)\theta}$ is the same, form a three-parameter subgroup of the four-parameter group $G_m$.*

A very important special case of $G_\infty$ remains to be noted, viz: when the invariant point $n$ is at infinity. All transformations of the group leave invariant the net of circles through the point at infinity. But this net of circles is the net of all straight lines in the plane. Thus the transformations of this group transform straight lines into straight lines; they are therefore projective transformations. These transformations retain the common property of all circular transformations that angular magnitudes are unchanged. The four-parameter group $G_\infty$ is therefore identical with the projective group of similitude, whose invariant figure is the line at infinity and the two circular points.

This result can also be shown analytically. Let $n = \infty$ in equation (3), whence we have

$$z_1 - m = k(z - m). \tag{7}$$

Equating real and imaginary parts we get

$$\begin{aligned} x_1 &= k'x - k''y - k'm' + k''m'' + m', \\ y_1 &= k''x + k'y - k'm'' - k''m' + m''. \end{aligned} \tag{8}$$

This is a projective transformation the vertices of whose invariant triangle are the point $(m', m'')$ and the two circular points.

The subgroups of $G_\infty$ give some interesting results. The path-curves of a one-parameter subgroup of loxodromic transformations are logarithmic spirals* around the point $m$, and the constant of the group $c$ in $k = e^{(c+i)\theta}$ is the cotangent of the angle between the curve and the radius vector. The path-curves of the one-parameter group of elliptic transformations are concentric circles about $m$; and the path-curves of a one-parameter group of hyperbolic transformations are straight lines through $m$.

Within $G_\infty$ all loxodromic transformations with constant $c$ form a three-parameter subgroup of logarithmic spiral motions with constant angle $\phi$. All elliptic transformations in $G_\infty$ form the three-parameter group of all rotations in the plane. All parabolic transformations in $G_\infty$ form the two-parameter group of all translations in the plane. Together all elliptic and all parabolic transformations in $G_\infty$ form the three-parameter group of all Euclidian motions in the plane. All hyperbolic transformations in $G_\infty$ form the three-parameter group of all *affine* transformations (*i. e.*, dilations) of the plane.

* Klein: Modulfunctionen, vol. 1, p. 168.

THEOREM 11. *All circular transformations leaving the point at infinity invariant are projective transformations, and the four-parameter group $G_\infty$ is identical with the four-parameter projective group in the plane whose invariant figure is the line at infinity and the two circular points.*

It seems to be a favorite method with Klein to express whenever possible projective groups in terms of the complex variable both in the plane and on the Neumann sphere; see for example Nicht-Euclidische Geometrie, vol. 2, page 184 ff and Höhere Geometrie, vol. 2, page 229 ff, and many other places.

We come now to the consideration of another group type of great importance. According to Lie's principle all transformations leaving a circle invariant form a group. Consider first a group of hyperbolic transformations leaving invariant $m$ and $n$ and every circle of the pencil through $m$ and $n$. Choose one of these circles $C$ and another point $n_1$ on $C$. The group of hyperbolic transformations with invariant points at $m$ and $n_1$ also leaves $C$ invariant. Thus all hyperbolic transformations having one invariant point at $m$ and the other also on $C$ leave $C$ invariant and form a two-parameter group. In this two-parameter group is included the one-parameter parabolic group whose invariant point is $m$ and whose invariant line is the tangent to $C$ at $m$. In like manner there is a two-parameter group for every point on $C$. All the transformations contained in these $\infty^1$ two-parameter groups form a three-parameter group leaving $C$ but no point on $C$ invariant. $\infty^2$ of these transformations are parabolic; these are distributed into $\infty^1$ one-parameter groups, but taken together do not form a two-parameter group.

There are also $\infty^3$ elliptic transformations which leave $C$ invariant. Let $m$ be any point within $C$ and $n$ its inverse point with respect to the circle $C$. The one-parameter group of elliptic transformations having its invariant points at $m$ and $n$ has $C$ among its pencil of invariant circles. In like manner all one-parameter groups of elliptic transformations whose invariant points are a pair of inverse points with respect to $C$ leave $C$ invariant. There are $\infty^2$ such pairs of points, and hence there are $\infty^3$ elliptic transformations in the group leaving $C$ invariant.

THEOREM 12. *There are $\infty^3$ circular transformations which leave invariant any given circle; these form a three-parameter group. This group is composed of all hyperbolic transformations whose invariant points are on the circle, of all elliptic transformations whose invariant points are a pair of inverse points with re-*

*spect to the circle, and of all parabolic transformations whose invariant point is on the circle and whose invariant line is a tangent to the circle at the invariant point.*

The transformations of this group $G_C$ are distributed into subgroups as follows: The elliptic transformations are distributed into $\infty^2$ one-parameter subgroups, but not into two-parameter subgroups. The hyperbolic transformations are distributed into $\infty^1$ two-parameter subgroups; each of these two-parameter groups break up into $\infty^1$ one-parameter subgroups, one of which is parabolic.

Since a straight line is considered as a circle through the point at infinity, it follows at once that there is a three-parameter group of transformations leaving a straight line invariant. This group $G_L$ is in all respects similar to the group $G_C$.

A very important special case of the group $G_L$ is when the line $L$ is the axis of real quantities.* $G_L$ then becomes the group of real projective transformations of the points on a real line. The properties of the real projective group are at once known from the properties of $G_C$.

THEOREM 13. *The three-parameter group of real projective transformations of the points on a line is a special subgroup of the six-parameter group of circular transformations of the points of the complex plane.*

There is still another type of three-parameter group consisting entirely of elliptic transformations which is closely related to the group $G_C$. This is the group of transformations of the complex plane which corresponds to the three-parameter group of rotations of a sphere about its centre. Every rotation of a sphere when projected stereographically upon the equatorial plane produces an elliptic transformation in that plane. Klein discusses the group of rotations of the sphere on pages 32–36 of his Ikosaeder, and on page 35 gives an analytic proof of the group property. The relation of this group in the plane to the group $G_C$ is shown as follows:

The invariant points of a one-parameter elliptic subgroup of $G_C$, being inverse points with respect to $C$, form a pair of corresponding points in a hyperbolic involution on the line joining the two points with the centre $O$ of the circle. The double points of the involution are the two points where the line cuts the circle. Every line through the centre of the circle $C$ is the bearer of such an involution; and all the

* Poincaré has investigated many of the properties of these groups $G_C$ and $G_L$ in his papers in *Acta Mathematica*, vols. 1 and 3, "Theorie des groupes fuchsiens" and "Theorie des groupes kleiniens."

one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions belong to the three-parameter group $G_c$. Let us consider a similar system of elliptic involutions on all lines of a pencil through $O$, such that the product of the distances from the centre of a pair of corresponding points is the same in all the involutions. Thus $OP \cdot OP' = -k^2$, constant for all the involutions. When a sphere is projected stereographically upon the equatorial plane, every pair of opposite points on the sphere project into a pair of corresponding points in one of these involutions. Thus all one-parameter groups of elliptic transformations whose invariant points are a pair of corresponding points in one of these involutions form a three-parameter group.

This three-parameter group leaves no figure of the plane invariant; but if it were allowable to use the language of projective geometry in speaking of the complex plane, we should say that this group leaves invariant an imaginary circle with centre at $O$ and radius equal to $ki$. We shall, therefore, designate this group as $G_{ic}$.

This completes the discussion of the subgroups of the general circular group. It remains to be shown that there are no other types of subgroups besides those discussed above. I shall attempt no formal proof, but shall only bring forward some general considerations bearing upon the question.

A circular transformation transforms points into points and circles into circles. We have considered all possible groups which leave invariant one or two points; a transformation leaving invariant more than two points is identical. We have also considered all possible groups of transformations leaving a circle invariant. If there be a continuous group characterized by the invariance of some curve other than a circle, such a curve must be the path curve of a one-parameter group. The only other path curve besides the circle is the double spiral of Holzmüller. This has two singular points and is invariant only under those transformations whose invariant points are these two singular points; hence, there is only one one-parameter group leaving invariant such a double spiral. These considerations indicate that there are no other subgroups of the general circular group.

Lie's theory of continuous groups based upon the infinitesimal transformation is better adapted than the method of this paper for determining the complete list of types of subgroups of a given group. It may be likened to a net which

gathers in its meshes all types of subgroups and lets none escape. My list of groups should be verified or corrected by the application of Lie's methods.

I append here a list of the group types discussed in the foregoing pages with a brief characterization of each.

(1) The six-parameter group $G_6$ of all circular transformations.

(2) The four-parameter group $G_4$ leaving a single point invariant.

(3) The two-parameter group $G_2$ of type $T$ leaving a pair of points invariant.

(4) The two-parameter group $G_2'$ of type $T'$ leaving a single point invariant.

(5) The one-parameter group $G_{1c}$ of type $T$ and constant $c$ in $k = e^{(c+i)\theta}$ leaving two points invariant. ($a$) The one-parameter group of elliptic transformations for which $c = 0$. ($b$) The one-parameter group of hyperbolic transformations for which $c = \infty$ and $\theta = 0$.

(6) The one-parameter parabolic group $G_1'$ with constant $\theta$ in $a = re^{i\theta}$ leaving a single point invariant

(7) The three-parameter group $G_{3c}$ of type $T$ and constant $c$ in $k = e^{(c+i)\theta}$ leaving a single point invariant.

(8) The three-parameter group $G_C$ of elliptic, hyperbolic and parabolic transformations leaving a circle invariant.

(9) The two-parameter group $G_{2C}$ of hyperbolic transformations leaving invariant a circle and a point on it.

(10) The three-parameter group of elliptic transformations $G_{1C}$.

The real projective transformations of the plane and of space may be treated in the same spirit and by the same methods here employed for the circular transformations. The writer hopes to be able in the near future to publish the full results of his investigations in these fields.

UNIVERSITY OF KANSAS,
*March* 6, 1897.

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## PLÜCKER'S COLLECTED PAPERS.

*Julius Plücker's gesammelte mathematische Abhandlungen.* Herausgegeben von A. SCHOENFLIES. Mit einem Bildniss Plücker's, und 73 in den text gedruckten Figuren. Leipzig, Druck und Verlag von B. G. Teubner. 1895. 8vo, pp. xxxvi + 620.

The Kgl. Gesellschaft der Wissenschaften zu Göttingen, of which Plücker was a corresponding member, recently under-

took the publication of his scientific papers, and appointed as editors Professors Schoenflies and Pockels. The volume whose separate title is given above is the first of the two which together form this collected edition. The only point in the arrangement that could be wished changed is the natural decision, referred to in the preface, not to reprint any of Plücker's longer works, inasmuch as they are still to be bought in the original editions. But interesting as these original editions may be, they are not satisfactory, for type and paper alike leave much to be desired; it is hardly possible, for instance, to read the *Theorie der algebraischen Curven* with any enjoyment; and so it is perhaps permissible to express the hope that at some future time, not too far distant, these also may appear in as readable a form as the *Collected Papers*.

This volume contains the thirty-nine memoirs by Plücker, and also Clebsch's "Gedächtnissrede", from the 16th volume* of the *Göttinger Abhandlungen*, a paper whose intrinsic value as the account given by one great geometer of the life-work of another can only increase with the lapse of time. It comes in here with special appropriateness, supplying the links between Plücker's work and that of his contemporaries, putting it into organic connection with the rest of the mathematical work of the period. This is a rôle for which Clebsch seems to have been peculiarly adapted by temperament, as witness one of his pupils, "von ihm lernten wir Anderen die Tendenz, auch fremde Untersuchungen umfassend in Betracht zu ziehen und mit den eigenen zu verweben"; and in the case of Plücker this sympathetic interpretation was specially needed, for his work produces an impression of aloofness—there is very little apparent consciousness of what the rest of the world might be doing. This intellectual isolation makes it easy to trace the workings of his mind all through the years in which he devoted himself to pure mathematics; we can see him generalizing his conceptions, as a preparation for a new stage in his work, making a clearance of unused materials after the pub-

* In the preface the reference for this notice is given as vol. 15, 1872; as a matter of fact it was read on the 2d of December, 1871, and is to be found in vol. 16, printed in 1872, but nevertheless the 1871 volume. The mistake is evidently due to the misprint on the title page of the reprint of the "Gedächtnissrede", and is to be found in various works, *e. g.*, in the text of the notice of Clebsch in vol. 7 of the *Mathematische Annalen*—though the correct reference is given in the bibliographical table at the end of this same article—and also on p. 35 of the second edition of Professor Gino Loria's "Il passato ed il presente delle principali teorie geometriche."

lication of a book, reaching out after a new geometry of space from the very first. He was not an erudite mathematician, for his mathematics came from within; he followed out his own train of thought, undistracted by the pressure of any ideas but his own, driven on by the "aufdrängende Fluth" of these, to which of deliberate purpose he yielded himself. A great part of the interest of the memoirs is in the view they give us of Plücker, possessed in turn by his various ideas, abridged notation, homogeneous coördinates, etc. This self-containedness early plunged him into difficulties in the matter of "plagiarism"; the first paper that he published after his Marburg dissertation, "Théorèmes et problèmes sur les contacts des sections coniques", *Annales de Mathématiques* (Montpellier), 1826, provoked a savage attack from Poncelet, due mainly to a misunderstanding arising from Gergonne's editorial alterations in and additions to the original memoir. These not unnaturally led Poncelet to believe that Plücker had worked by the unacknowledged light of his own Proprietés projectives, published in 1822; this misapprehension accounts for his irritation, though it does not justify the expression he gave to it. This protest, and Plücker's temperate rejoinder,* are here given by the editor, and with good reason, for apart from their historical interest they help us to understand Plücker's attitude towards his contemporaries. The accusation of plagiarism might, with a little ingenuity, be brought against many creative mathematicians; when the same field is occupied simultaneously by several, even though they be of different nationalities, and totally different intellectual bias and equipment, coincidences of results are certain to occur, the more frequently in proportion to the actual independence of the workers; for the plagiarist of ideas does not openly label his work by plagiarizing results also. Different as was Plücker's line of thought from that of his contemporaries, he evidently determined to be on the safe side; so he inserted, even impatiently, references to Poncelet wherever he detected any likeness in the results.

But the march of his ideas is his best defence. Every one of his books develops some particular conception, original with him, even though it may have occurred also to his

* The editor of the *Bulletin des Sciences de Férussac*, in vol. 9 of which Plücker's reply appeared, printed it with the remark that he had suppressed "quelques épithètes inutiles au succès de la discussion," so possibly the rejoinder was not really so temperate as it now appears.

Plücker refers to the matter again in the preface to the second volume of the *Entwicklungen*, p. vi.

contemporaries; and the first germs and applications of each conception are to be found in the memoirs immediately preceding the issue of the volume. On account of this intimate connection between the memoirs and the longer works, one is inclined to wish that the names of the successive volumes had been inserted in their proper places in the table of contents, and that all through this table the dates had been supplied.

In memoirs Nos. 4 and 5, 1827, 1828, abridged notation is used; to the explanation of this, and its application to conics, vol. 1 of the *Analytisch-geometrische Entwicklungen*, 1828, is devoted. The 5th paper, "Recherches sur les courbes algébriques de tous les degrés", is fundamental in the theory of the intersections of curves; in the 6th the argument is applied to surfaces; Nos. 7 and 8 also relate to solid geometry. Nos. 9, 10 and 11 deal respectively with homogeneous coördinates, abridged notation, and line coördinates, all of which are expounded in vol. 2 of the *Entwicklungen*, 1831; in the closing words of the preface we see Plücker already under the influence of new ideas, destined to find their expression in his next volume, the *System der analytischen Geometrie*, 1835. Of the intervening papers, No. 12 contains a number of enunciations of theorems and problems; Nos. 13–19 seem to have been an emptying of his note-books, whose miscellaneous contents he cast into the form of "Analytisch-geometrische Aphorismen"; Nos. 20 and 21 treat of the foci of curves, and of the relations connecting the order and class of a curve with the numbers of its point and line singularities. These two were written shortly after the time, the summer of 1832, at which, by Plücker's own account, the principal ideas of the *System* came into his mind; the kernel of the second of these papers is the precise description of the two-fold generation of a curve. In the preface to the *System*, after referring to his "Vorliebe" for line coördinates, he insists at some length on this dualistic view, and its expression by means of point and line coördinates, and points out that the passage from one mode of representation to the other involves a discontinuity. This statement is of course exact in the very special sense in which it was first intended, as referring to any singularity on the curve, and it is a part of Plücker's renown to have discriminated so clearly between point singularities and line singularities. But though the theories of point and line coördinates depend on two different conventions as to the nature of a geometrical figure—conventions which primarily are mutually exclusive, whose

combination appears at first sight to result in arguing in a circle—yet it seems an unnecessary restriction on the generality and elasticity of the symbolism to employ exclusively one set of coördinates or the other in any given investigation. The statement in the first chapter of Salmon's Higher Plane Curves (Cayley) "there is little occasion for any explicit use of line coördinates" seems distinctly unfortunate; for the full advantage of the duality between point and line coördinates is obtained only when we hold ourselves at liberty to use both systems at once, representing points and lines indifferently by coördinates or equations. The objection that this is using the conventions in an unjustifiable manner, relying on two different ones simultaneously, loses its force when we reflect that neither convention alone expresses the whole truth, and that a judicious misuse of conventions is frequently essential to progress.

The *System* contains also Plücker's discussion of cubic curves, analyzed by Cayley in his memoir "On the classification of cubic curves", 1864 (Collected Mathematical Papers, V, No. 350); additional theorems on cubics are given in No. 27, 1847. Practically continuous with the *System* is the 22d paper, 1836, "Enumération des courbes du quatrième ordre, d'après la nature différente de leurs branches infinies"; this and the next two were published during the preparation of the *Theorie der algebraischen Curven*, 1839, in which Plücker applied his method of abridged notation to the discussion of asymptotes and of singular points and lines, with respect to quartics, making use of the method of counting constants. With the *Theorie* he regarded his work in plane analytical geometry as finished, "liegt der Cyclus meiner Arbeiten im Gebiete der analytischen Geometrie vollständig vor", and he turned to the geometry of space, a subject frequently referred to in his earlier writings; in 1846 he published the *System der Geometrie des Raumes*, in which is already to be found the germ of his later line-geometry. Only one paper appeared in this interval, No. 26, "Aphorismen aus der Geometrie des Raumes", 1842.

He now resolved to give up his mathematical researches, to which he had devoted more than twenty years, feeling that his ideas, as to whose value he himself had no misgivings, did not meet with due recognition from his contemporaries; but before renouncing geometry, as he supposed for ever, he arranged for publication the investigations he had in hand, so that Nos. 27–32 all appeared in 1847.

For nearly twenty years Plücker devoted himself to physics, and accordingly this volume contains no memoirs

between 1847 and 1865. In 1865, however, encouraged by the interest in his researches manifested by many English mathematicians, in the forefront of whom must be named Cayley, he returned to the subject of the geometry of space. Referring to his abandonment of his geometrical work twenty years before, he said that while the details had escaped from his memory, two leading questions had remained dormant in his mind; the first was that of introducing right lines as elements of space, instead of points and planes; and in close connection with this was the second, a question in mechanics. To these he devoted the last three years of his life, developing his ideas in a series of papers, Nos. 33–38, published in England, France and Italy, all leading up to his *Neue Geometrie des Raumes*, 1868–9.

The last paper in the volume, No. 39, published in 1833, is a preliminary account of the *System der analytischen Geometrie*, on which he was then engaged. It is sometimes remarked that the eminently proper person to review a book is the writer thereof, and in this announcement, as in his prefaces, Plücker appears as his own sympathetic reviewer.

The concluding thirty pages of the volume are devoted to the editorial comments; by his interesting historical notes and helpful criticisms here contained, as well as by the care bestowed in making good the original carelessness in the matter of proof-reading, Professor Schoenflies has made all lovers of Plücker his debtors.

CHARLOTTE ANGAS SCOTT.

BRYN MAWR COLLEGE,
*October*, 1897.

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## NOTES.

PROFESSOR NEWCOMB has chosen the philosophy of hyperspace as the subject of his presidential address to be delivered at the annual meeting of the AMERICAN MATHEMATICAL SOCIETY on December 29th.

A NEW list of members of the AMERICAN MATHEMATICAL SOCIETY will be published in January. Blank forms for furnishing necessary information have been sent to each member, and a full and prompt response is requested.

AT the annual meeting of the London Mathematical Society, held on November 11, the following officers and members of the Council were elected for the ensuing year: President, Professor E. B. Elliott; Vice-Presidents, Major

P. A. MacMahon and Dr. E. W. Hobson; Treasurer, Dr. J. Larmor; Secretaries, R. Tucker and A. E. H. Love. Other members of the Council, Lieutenant-Colonel A. J. Cunningham, Dr. J. W. L. Glaisher, Professor M. J. M. Hill, Professor W. H. H. Hudson, Morgan Jenkins, A. B. Kempe, F. S. MacAulay, D. B. Mair, G. B. Mathews and W. D. Niven.

THE department of mathematics at Wellesley College has been divided, Miss ELLEN HAYES, hitherto professor of mathematics, becoming professor of applied mathematics. Miss E. L. BURRELL has been appointed acting professor of pure mathematics. Miss H. A. MERRILL, Miss B. DENIS and Miss H. M. KELSEY have been appointed instructors in mathematics.

AT the University of Chicago Dr. J. W. A. YOUNG has been advanced to an assistant professorship of mathematical pedagogy, and Dr. HARRIS HANCOCK and Mr. HERBERT E. SLAUGHT have been advanced to instructorships in mathematics.

DR. R. J. ALEY, who was last year at the University of Pennsylvania as fellow in mathematics, has resumed his duties as professor of mathematics at Indiana University. Assistant Professor D. A. ROTHROCK, of the same department, has been granted leave of absence during the present year and is studying at Leipzig.

MR. JOSEPH ALLEN, formerly instructor in mathematics at Cornell University, has been appointed tutor in mathematics at the College of the City of New York.

ALEXANDER PELL, Ph. D. (Johns Hopkins, 1897), has been appointed professor of mathematics and astronomy at the University of South Dakota.

DR. LOTHAR HEFFTER, associate professor of mathematics at the University of Giessen, has been appointed to a similar position at the University of Bonn.

DR. ARNOLD SOMMERFELD, lecturer at the University of Göttingen, has been called to the professorship of mathematics at the Clausthal Academy, to succeed Professor FRANZ MEYER, who is now at Königsberg.

PARIS FACULTY OF SCIENCES. The mathematical courses offered during the first semester of the current academic year are the following:—Professor G. DARBOUX: Theory of curvilinear coördinates.—Professor E. GOURSAT: Differential and integral calculus from the point of view of real

variables.—Professor P. APPELL: General laws of equilibrium and motion.—Professor H. POINCARÉ: Planetary perturbations and the development of the perturbative function.—Professor V. J. BOUSSINESQ: Analytical theory of heat.—Professor G. KŒNIGS: Kinematics of a solid body; motions depending on several parameters; kinematics of deformable bodies.—Mr. RAFFY: Mathematics introductory to the study of physics.—Mr. ANDOYER: Motion of rotation of heavenly bodies by the method of variation of arbitrary constants. In addition, mathematical conferences are conducted by Messrs. Hadamard, Puiseux, Andoyer and Blutel.

For the second semester the following courses have been announced:—Professor PICARD: Theory of algebraic surfaces.—Professor GOURSAT: Principles of the theory of analytical functions.—Professor BOUSSINESQ: Propagation of heat in crystals.—Professor KŒNIGS: Study of Machines. Professor Appell will also lecture, but the subject of his course has not been announced.

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## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

ABRAMOV (V.). Sketch of the methods of integrating differential equations. Part I. (In Russian.) Moscow, 1897. 8vo. 24 pp. Mk. 1.60

ARCHIMEDES. Works, edited in modern notation, with introductory chapters, by T. L. Heath. Cambridge University Press, 1897. 8vo. 514 pp. 15s.

BOOLE (M. E.). See GRATRY.

BURALI-FORTI (C.). Introduction à la géométrie différentielle, suivant la méthode de H. Grassmann. Paris, Gauthier-Villars, 1897. 8vo. 11 and 166 pp. Fr. 4.50

BURNSIDE (W. S.). The theory of groups of a finite order. Cambridge University Press (New York, Macmillan), 1897. 8vo. 16 and 388 pp. Cloth. $3.75

CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices du ministre de l'instruction publique. Première série. Vol. X: Extraits des comptes rendus de l'Académie des sciences. Paris, Gauthier-Villars, 1897. 4to. Fr. 25.00

DELASSUS (E.). Leçons sur la théorie analytique des équations aux dérivées partielles du premier ordre. Paris, 1897. 8vo. 88 pp. Fr. 5.00

DIRICHLET. See LEJEUNE-DIRICHLET.

ECHEGARAY (J.). Lecciones sobre la resolucion de las ecuaciones y la teoria de Galois. Madrid, 1897. 4to. 272 pp. Fr. 13.00

FISHER (I.). A brief introduction to the infinitesimal calculus, designed especially to aid in reading mathemathical economics and statistics. New York, Macmillan, 1897. 16mo. 7 and 84 pp. Cloth. $0.75

FREYCINET (C.). Philosophy of mathema'ics. Translated by V. Obrèimov. (In Russian.) St. Petersburg, 1897. 8vo. 166 pp. Mk. 2.50

FRICKE (R.) und KLEIN (F.). Vorlesungen über die Theorie der automorphen Funktionen. Vol. 1: Die gruppentheoretischen Grundlagen. Leipzig, Teubner, 1897. 8vo. 16 and 634 pp. Mk. 22.00

FUCHS (L.). See LEJEUNE-DIRICHLET (G.).

FUCHS (R.) Ueber die Periodicitätsmoduln der hyperelliptischen Integrale als Funktionen eines Verzweigungspunktes. [Diss.] Berlin, 1897. 4 to 24 pp.

FURTWÄNGLER (P.). See KLEIN (F.).

GRATRY and BOOLE. Mathematical psychology translated from language of higher calculus into that of elementary geometry, by Mary Everest Boole. London, Sonnenschein, 1897. 12mo. 124 pp. 3s.

HALSTED (G. B.). See LOBACHEVSKI.

HEATH (T. L.). See ARCHIMEDES.

HERMITE (C.). See LAGUERRE.

HUTCHINSON (J. I.). A special form of a quartic surface. (*Annals of Mathematics*, vol. 11, pp. 158–160.) 4to.

KLEIN (F.). Vorlesungen über ausgewahlte Kapitel der Zahlentheorie. Vol. 1: Vorlesung im Wintersemester 1895–96, ausgearbeitet von A. Sommerfeld. 5 and 391 pp. Vol. 2: Vorlesung im Sommersemester 1896, ausgearbeitet von A. Sommerfeld und P. Furtwängler. 5 and 354 pp. (Autogr.) Leipzig, Teubner, 1897. 4to. Mk. 14.50

KLEIN (F.). See FRICKE (R.).

KOMMERELL (K.). Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen. [Diss.] Tübingen, 1897. 8vo. 53 pp.

LAGUERRE. Oeuvres, publiées sous les auspices de l'Académie des sciences, par C. Hermite, H. Poincaré et E. Rouché. (En 2 volumes.) Vol. I: Algèbre; calcul intégral. Paris, Gauthier-Villars, 1898. 8vo. Fr. 15.00

LAKHTIN (L. K.). Differential resolvents of algebraic equations. (In Russian.) Moscow, 1897. 8vo. 420 pp., 2 plates. Mk. 14.00

LAMB (H.). An elementary course of infinitesimal calculus. Cambridge University Press (New York, Macmillan), 1897. 12mo. 20 and 616 pp. Cloth. $3.00

LEJEUNE-DIRICHLET (G.) Werke, herausgegeben auf Veranlassung der Königl. preussischen Akademie der Wissenschaften von L. Kronecker; fortgesetzt von L. Fuchs. (In 2 Bänden.) Vol. II. Berlin, Reimer, 1897. 4to. 10 and 422 pp. Mk. 18.00

LOBACHÉVSKI (N. I.). [Introduction to the] New principles of geometry, with complete theory of parallels. Translated from the Russian by G. B. Halsted. Volume fifth of the neomonic series. Austin, Texas, 1897. 8vo. 27 pp.

MACDONALD (W. J.). Higher geometry; introduction to modern geometry and elementary geometrical conics. 2d edition. London, Thin, 1897. 12mo. 198 pp. 3s. 6d.

MEYER (A.). Laerebog i Infinitesimalregningens Begyndelsesgrunde. Kjöbenhavn, 1897. 8vo. 140 pp. Mk. 3.80

MILLER (G. A.). On the primitive substitution groups of degree fifteen. (*Proceedings of the London Mathematical Society*, Vol. 28, No. 608.) 8vo. pp. 533–544.

MÜLLER (F.). Der Esslinger Mathematiker Michael Stifel. [Progr.] Esslingen, 1897. 4to. 39 pp.

NEKRÀSOV (P. A.). Application of algebra to geometry; concrete presentation of Lobachevsky's geometry. 2d edition. (In Russian.) Moscow, 1897. 8vo. 272 pp. Mk. 7.50

OBRÈIMOV (V.). See FREYCINET (C.).

PAGE (J. M.). Ordinary differential equations; an elementary textbook, with an introduction to Lie's theory of the group of one-parameter. New York, Macmillan, 1897. 12mo. 18 and 226 pp. Cloth. $1.25

PAMPUCH. Das allgemeine Malfattische Problem nebst fünf demselben äquivalenten Aufgaben. [Progr.] Strassburg, 1897. 4to. 40 pp., 2 plates.

PEANO (G.). Logique mathématique. (Formulaire de mathématiques, Tome II, § 1.) Turin, Bocca, 1897. 8vo. 63 pp.

POINCARÉ (H.). See LAGUERRE.

RICCI (G.). Lezioni sulla teoria delle superficie. Verona e Padova, Drucker, 1897. 8vo. 8 and 416 pp. (Lithogr.) Fr. 10.00

ROUCHÉ (E.). See LAGUERRE.

SCHEFFLER (H.). Vermischte mathematische Schriften, enthaltend: (1) Zusätze zur Theorie der Gleichungen; (2) Die quadratische Zerfallung der Zahlen; (3) Die Phönixzahlen. Braunschweig, Wagner, 1897. 8vo. 109 pp. Mk. 2.00

SOKHOTSKY (I.). On the expansion of definite integrals and functional equations in series. (In Russian.) St. Petersburg, 1897. 8vo. 73 pp.

SOMMERFELD (A.). See KLEIN (F.).

THOMA (H.). Ein Problem aus der Variationsrechnung. [Progr.] Strassburg, 1897. 4to. 19 pp.

VERONESE (G.). Sul postulato della continuità. (*Rendiconti della R. Accademia dei Lincei*, Vol. 6, 2do sem., serie 5a, fasc. 6to.) Roma, 1897. 4to. Pp. 161–168.

VORONOI (G.). On a generalization of the algorithms of continued fractions. (In Russian.) Warsaw, 1897. 8vo. 219 pp. Mk. 4.00

WINSTON (M. F.). Ueber den Hermite'schen Fall der Lamé'schen Differentialgleichung. [Diss.] Göttingen, 1897. 8vo. 63 pp. 17 plates.

WÖLFING (E.). Die singulären Punkte der Flächen. [Habilitationsschrift.] Dresden, Teubner, 1896. 8vo. 25 pp.

## II. ELEMENTARY MATHEMATICS.

ANGELI (A. DE). Sui programi di matematica per le scuole normali e complementari. Mondovi, 1897. 8vo. 88 pp. Fr. 3.00

BEMAN (W. W.). See KLEIN (F.).

BENEDETTI (G.). Scienza dei triangoli Pitagorici dipendente dai tre mezzi determinati delle proporzioni geometriche di questa scienza di cui sono una parte integrale. [Progr.] Trieste, 1897. 8vo. 21 pp.

BLASCHKE (J.). Ein Beitrag zur elementaren Behandlung der Kegelschnitte. [Progr.] Graz, 1897. 8vo. 12 pp., 1 plate.

BONACCORSI (G.). La determinazione numerica degli elementi incogniti di un triangolo. Forlì, Mar ani, 1897. 12mo. 22 pp., 1 plate. Fr. 0.70

EUCLIDE. Gli elementi, nuovamente tradotti, con note, aggiunte ed esercizî, ad uso dei ginnasî e dei licei, per cura di A. Socci e G. Tolomei. Lib. V–VI. Firenze, Le Monnier, 1897. 8vo. pp. 287–480. Fr. 1.50.

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 52ste Auflage. Halle, 1897. 8vo. 166 and 35 pp. Half leather. Mk. 2.50

GIAVARINI (C.). A difesa del mio primo studio: Trisezione elementare dell' angolo piano rettilineo, impresso in Modena il 12 ottobre 1892. Modena, Rossi, 1897. 8vo. 15 pp.

HAENTZSCHEL (E.). Ueber die verschiedenen Grundlegungen in der Trigonometrie; eine historisch-kritische Studie. Leipzig, Dürr, 1897. 8vo. 8 pp. Mk. 0.40

KEIGWIN (H. W.). The elements of geometry. New York, Holt, 1897. 12mo. 6 and 227 pp. $1.00

KIRCHHOFF (E.). Anleitung zur Erteilung des Unterrichts in der Raumlehre; nebst einem Anhang, enthaltend die Resultate zu den Schülerheften. 3te Auflage. Leipzig, Hirt, 1897. 8vo. 2 and 64 pp. Mk. 0.60

KLEIN (F.). Famous problems of elementary geometry: the duplication of the cube, the trisection of an angle, the quadrature of the circle. An authorized translation of F. Klein's Vorträge über ausgewählte Fragen der Elementargeometrie, ausgearbeitet von F. Tägert, by W. W. Beman and D. E. Smith. Boston, Ginn, 1897. 12mo. 10 and 80 pp. Cloth. $0.50

LANE (F. O.). A junior algebra. London, Holden, 1897. 12mo. 232 pp. 1s. 6d.

———. A junior algebra; with answers. London, Holden, 1897. 12mo. 268 pp. 2s.

LIEBER (H.) und MÜSEBECK (C.). Aufgaben über kubische und diophantische Gleichungen, Determinanten und Kettenbrüche, Kombinationslehre und höhere Reihen. Berlin, Simson, 1897. 8vo. 5 and 129 pp. Mk. 2.40

MUCH (A.). See REIDT (F.).

MÜLLER (C. A.). Multiplicationstabellen, auch für Divisionen anwendbar; bearbeitet nach einer neuen Anordnung. Karlsruhe, Braun, 1897. 8vo. 8 and 201 pp., 1 table. Cloth. Mk. 3.00

MÜSEBECK (C.) See LIEBER (H.).

NOETLING (W.). Elements of constructive geometry instructively presented; from the German of K. H. Stöcker. Boston, Silver, Burdett & Co., 1897. 12mo. 3 and 62 pp. Cloth. $0.36

O'DEA (J. J.). New explicit algebra in theory and practice. For teachers, university students, etc. Part I. London, Longmans, 1897. 12mo. 242 pp. 3s.

PRZHEVALSKY (I.). Collection of problems in algebra. 6th edition. (In Russian.) Moscow, 1897. 8vo. 272 pp. Mk. 7.50

REIDT (F.). Sammlung von Aufgaben und Beispielen aus der Trigonometrie und Stereometrie. Teil II: Stereometrie. 4te Auflage. Neu bearbeitet von A. Much. Leipzig, Teubner, 1897. 8vo. 8 and 194 pp. Mk. 3.00

———. Stereometrie; Resultate der Rechnungsaufgaben. Leipzig, Teubner, 1897. 8vo. 58 pp. Mk. 1.00

SACHS (J.). Lehrbuch der ebenen Elementar-Geometrie (Planimetrie). Teil 8: Die Anwendung der Aehnlichkeit auf die Lehre vom Kreis. Bearbeitet nach System Kleyer. Stuttgart, Maier, 1897. 8vo. 8 and 226 pp., 1 table. Mk. 5.00

———. Tabelle der regelmässigen Vielecke. (Aus "Lehrbuch der ebenen Elementar-Geometrie.") Stuttgart, Maier, 1897. Folio. Mk. 0.50

SCHLÖMILCH (O.). Fünfstellige logarithmische und trigonometrische Tafeln. 14te Auflage. Wohlfeile Schulausgabe. Braunschweig, Vieweg, 1897. 8vo. 3 and 151 pp. Mk. 1.00

SCHRÖN (L.). Siebenstellige gemeine Logarithmen der Zahlen von 1 bis 108 000 und der Sinus, Cosinus, Tangenten und Cotangenten aller Winkel des Quadranten von 10 zu 10 Secunden, nebst Interpolationstafel zur Berechnung der Proportionalteile. 23ste revidirte Ausgabe. (In 3 Teilen.) Braunschweig, Vieweg, 1897. 8vo. Mk. 6.00

SJÖGREN (F. A.). Instrument for Cirkelbagars Rectification och Delning, Cirkelsektorers och Segments Kvadrering, etc. Stockholm, 1896. 8vo. 18 pp. Mk. 1.20

SMITH (D. E.). See KLEIN (F.).

SOCCI (A.). See EUCLIDE.

STÖCKER (K. H.). See NOETLING (W.).

TÄGERT (F.). See KLEIN (F.).

TOLOMEI (G.). See EUCLIDE.

## III. APPLIED MATHEMATICS.

ANGLÈS (R.). Traité du quadrant. (Montpellier, 13me siècle.) Texte latin et ancienne traduction grecque, publiés par P. Tannery. Paris, 1897. 4to. 84 pp.

ANTOMARI (X.). Traité de géométrie descriptive, à l'usage des élèves de mathématiques élémentaires, des aspirants au baccalauréat de l'enseignement classique (2e série) et au baccalauréat de l'enseignement moderne (3e série), et des candidats à l'Institut agronomique. Paris, Nony, 1898. 8vo. 164 pp.

BACON (N. T.). See COURNOT (A.).

BIGGS (C. H. W.). First principles of electricity and magnetism. London, Biggs, 1897. 12mo. 498 pp. 3s. 6d.

BOLTZMANN (L.). Vorlesungen über die Principe der Mechanik. (In 3 Teilen.) Teil I, enthaltend die Principe, bei denen nicht Aus-

drücke nach der Zeit integrirt werden, welche Variationen der Coordinaten oder ihrer Ableitungen nach der Zeit enthalten. Leipzig, Barth, 1897. 8vo. 10 and 241 pp. Mk. 6.00

BRIGGS (W.). and BRYAN (G. H.). An elementary text-book of mechanics. 2d edition. (University tutorial Series.) London, Clive, 1897. 8vo. 344 pp. 3s. 6d.

BRYAN (G. H.). See BRIGGS (W.).

CHAILAN (E.) Géométrie descriptive, à l'usage des candidats au baccalauréat (enseignment classique, 2e partie, 2e série ; enseignement moderne, 1e partie et 2e partie, 3e série) et des candidats à l'Institut agronomique. Paris, Poussielgue, 1897. 12mo. 192 pp.

COURNOT (A.). Researches into the mathematical theory of wealth (1838). Translated by N. T. Bacon, with a bibliography of mathematical economics by J. Fischer. New York, Macmillan, 1897. 16mo. 9 and 213 pp. $0.75

DOBBS (W. J.). Elementary geometrical statics; an introduction to graphic statics. London and New York, Macmillan, 1897. 12mo. 11 and 340 pp. Cloth. $1.80

DUHEM (P.). Traité élémentaire de mécanique chimique fondée sur la thermodynamique. Vol. II: Vaporisation et modifications analogues ; continuité entre l'état liquide et l'état gazeux; dissociation des gaz parfaits. Paris, Hermann, 1898. 8vo. 378 pp.

ERNST (C.). Eine Theorie des elektrischen Stromes auf Grund des Energieprincipes. München, Lüneburg, 1897. 8vo. 64 pp. Mk. 2.00

FISHER (I.). See COURNOT (A.).

GÉRARD (E.). Leçons sur l'électricité ; professées à l'Institut électrotechnique Montefiore annexé à l'Université de Liége. (En 2 volumes.) Vol. II: Canalisation et distribution de l'énergie électrique ; applications de l'électricité à la téléphonie, à la télégraphie, à la production et à la transmission de la puissance motrice, à la traction, à l'éclairage et à la métallurgie. 5e édition. Paris, Gauthier-Villars, 1898. 8vo. Fr. 12.00

GRÜNWALD (E.). Die Herstellung und Verwendung der Accumulatoren in Theorie und Praxis. Ein Leitfaden. 2te Auflage. Halle, Knapp, 1897. 12mo. 6 and 154 pp. Mk. 3.00

GÜMBEL (L.). Das Stabilitätsproblem des Schiffsbaues. Berlin, Siemens, 1897. 8vo. 6 and 49 pp., 6 plates. Mk. 2.40.

HANSSEN (C. G. J.). Reform of chemical and physical calculations. London and New York, Spon & Chamberlain, 1897. 4to. 73 pp. Cloth. $2.60

HAY (A.). Principles of alternate current working. London, Biggs, 1897. 12mo. 296 pp. 5s.

JAMIESON (A.). A text-book on applied mechanics, for engineering students. Vol. 2. London, Griffin, 1897. 12mo. 402 pp. 7s. 6d.

LEUSCHNER (A. O.). Beiträge zur Kometenbahnbestimmung. [Diss.] Berlin, 1897. 4to. 46 pp., 1 plate.

LOVE (A. E. H.). Theoretical mechanics. Introductory treatise on the principles of dynamics, with applications and examples. Cambridge University Press, 1897. 8vo. 396 pp. Cloth. 12s.

MASI (F.) Le nuove vedute nelle ricerche teoriche ed esperimentali sull' attrito ; determinazione del valore industriale dei lubrificanti. Bologna, Zanichelli, 1897. 8vo. 131 pp., 1 plate.

MERRIMAN (M.). The method of least squares. 7th edition. New York, Wiley, 1897. 8vo. 236 pp. Cloth. $2.00

MOSHAMMER (K.). Hydromechanik. Lehrtext zum Gebrauche an höheren Gewerbeschulen und Buch zum Selbststudium. Wien, Deuticke, 1897. 8vo. 73 pp. Mk. 2.00

OPPENHEIM (S.). Zur Lehre von der Centralbewegung in elementarer Darstellung. [Progr.] Arnau, 1897. 8vo. 24 pp.

PERRY (J.). Applied mechanics. Treatise for students who have time to work experimental, numerical and graphical exercises. London, Cassell, 1897. 12mo. 686 pp. 7s. 6d.

ROSENBERG (F.). First stage mechanics of solids. For the elementary examination of the Science and art department. (Organized science series.) 2d edition. London, Clive, 1897. 12mo. 312 pp. Cloth. 2s.

SMITH (J. H.) An elementary treatise on the metric system of weights and measures. London, Longmans, 1897. 12mo. 112 pp. 2s.

TANNERY (P.). See ANGLÈS (R.).

THOMPSON (S. P.) Elementare Vorlesungen über Elektricität und Magnetismus. Deutsch auf Grund der neuesten Auflage des Originals von A. Himstedt. 2te Auflage. Tübingen, Laupp. 1897. 8vo. 8 and 604 pp. Mk. 7.00

ZEISSIG (C.). Ein einfacher Fall der transversalen Schwingungen einer rechteckigen elastischen Platte. [Diss.] Göttingen, 1897. 8vo. 52 pp., 2 plates.